1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\ddot{u}^2 + \dot{u}^2\right)\,dx.$$

(a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) = u'(0) = 1$.

(b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

2. Let us consider the boundary value problem

$$u''(x) = \frac{1 + e^{u(x)}}{1 + e^{u'(x)}}, \qquad u(0) = 3, \quad u(3) = 0.$$

(a) Discuss existence, uniqueness and regularity of the solution.

(b) Prove that $u'(0) < -1$.

See CdV 2019_3, Ex 1 and Ex 2.

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3. Let Ω be a ball in $\mathbb{R}^3$. For every positive integer m, let us set

$$\sup\left\{\int_\Omega |u-\arctan(xyz)|^m\,dx\,dy\,dz : u \in C^1(\Omega),\ \int_\Omega (u^2+2u_x^2+3u_y^2+4u_z^2)\,dx\,dy\,dz \le 5\right\}.$$

(a) Prove that in the case $m=3$ the supremum is actually a maximum, at least in the larger class $H^1(\Omega)$.

(b) Determine all positive integers m such that the supremum is a real number.

(a) Since the embedding $H^1(\Omega) \to L^3(\Omega)$ is compact, we can apply the standard direct method. Just note that $|u-\arctan(xyz)|^3$ is continuous and the constraint is closed under weak convergence.

A straightforward (but long) application of Lagrange mult. and reg. theory shows that maximizers are actually smooth.

(b) The supremum is finite if and only if $\boxed{m \le 6}$, namely if and only if the embedding $H^1 \to L^q$ takes place.

If $m \le 6$ it turns out that

$$\|u-\arctan(xyz)\|_{L^m} \le \|u\|_{L^m} + \|\arctan(xyz)\|_{L^m}$$

$$\le C_1 \|u\|_{1,2,\Omega} + C_2$$

Sobolev ineq.

bounded because of the given condition

If $m>6$, then there exists $u \in H^1(\Omega)\setminus L^m(\Omega)$. Up to multiplying by a small constant, we can also assume that the condition $\ldots \le 5$ is satisfied. On the other hand, also $u-\arctan(xyz) \notin L^m$, and therefore the supremum in the class $u \in H^1$ with $\ldots \le 5$ is $+\infty$.

With an approximation procedure we can show that the supremum is $+\infty$ also in $C^1(\Omega)$.

An explicit example of sequence that realizes the supremum is

$$u_m(x) = \frac{\varepsilon}{\|x-x_0\|^{3/m} + \frac{1}{m}}$$

where x_0 is the center of the ball, and $\varepsilon > 0$ is small enough.

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4. For every $f : (0,5) \to \mathbb{R}$, let us set

$$[Tf](x) := \sin(f(x)).$$

Determine whether the restriction of T defines

(a) a *continuous* mapping $L^8((0,5)) \to L^1((0,5))$,
(b) a *continuous* mapping $L^1((0,5)) \to L^8((0,5))$,
(c) a *compact* mapping $H^8((0,5)) \to L^{2019}((0,5))$.

(a) $\boxed{\text{YES}}$ $\quad \|Tf - Tg\|_{L^1} = \int_0^5 |\sin(f(x)) - \sin(g(x))|$

$$\leq \int_0^5 |f(x)-g(x)| = \int_0^5 \underset{\frac{7}{8}}{1} \cdot \underset{\frac{1}{8}}{|f(x)-g(x)|} \leq 5^{7/8} \cdot \left\{ \int_0^5 |f(x)-g(x)|^8 \right\}^{1/8}$$

$= c \cdot \|f-g\|_{L^\infty} \implies T$ is Lipschitz continuous.

(b) $\boxed{\text{YES}}$ $\quad \|Tf - Tg\|_{L^8}^8 = \int_0^5 |\sin(f(x)) - \sin(g(x))|^8$

$$= \int_0^5 \underbrace{|\ldots|^7}_{\leq 2^7} \cdot |\ldots| \leq 2^7 \int_0^5 |\sin(f(x)) - \sin(g(x))|$$

$$\leq 2^7 \int_0^5 |f(x)-g(x)| = 2^7 \|f-g\|_{L^1}$$

$\implies T$ is $\frac{1}{8}$ - Hölder continuous.

(c) $\boxed{\text{YES}}$ If $\{f_n\} \subseteq H^8$ is a bounded sequence, then there exists a subsequence $f_{n_k} \to f_\infty$ uniformly, because of the compact imbedding of H^8 in C^7.
It follows that $\sin(f_{n_k}) \to \sin(f_\infty)$ uniformly, and in particular

$$Tf_{n_k} \to Tf_\infty \quad \text{in } L^{2019}.$$

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<u>Remark</u> The restriction of T is continuous as a function

$$L^p((a,b)) \to L^q((a,b))$$

for every choice of the $\boxed{\text{REAL}}$ exponents $p \geq 1$ and $q \geq 1$ (note that $q = +\infty$ is NOT allowed). Hint:

$f_n \to f_\infty$ in $L^p \implies f_{n_k}(x) \to f_\infty(x)$ for a.e. $x \in (a,b)$

$\implies \sin(f_{n_k}(x)) \to \sin(f_\infty(x))$ for a.e. $x \in (a,b) \implies \ldots$