Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 01 January 2019

1. (a) Let Ω be the unit ball in $\mathbb{R}^2$ with center in the origin. Determine whether

$$\inf\left\{\int_\Omega \sinh u\,dx\,dy : u \in C_c^\infty(\Omega),\ \int_\Omega \left(u_x^6 + u_y^4\right) dx\,dy \leq 3\right\}$$

is a real number.

(b) Let Ω be the unit ball in $\mathbb{R}^{2019}$ with center in the origin. Determine whether

$$\inf\left\{\int_\Omega \sinh u\,dx : u \in C_c^\infty(\Omega),\ \int_\Omega |\nabla u|^2\,dx \leq 3\right\}$$

is a real number.

2. The "new year norm" on $\mathbb{R}^2$ is defined by

$$\|(x,y)\| := 2018|x| + 2019|y|.$$

(a) Determine all aligned functionals of $(0,1)$ with respect to this norm.

(b) Determine all aligned functionals of $(-2,3)$ with respect to this norm.

(c) Determine the norm of the identity as an operator from $\mathbb{R}^2$ with the "new year norm" to $\mathbb{R}^2$ with the Euclidean norm.

3. For every function $f : (0,1) \to \mathbb{R}$, and every real number $a > 0$, let us set

$$[T_a f](x) := f(x^a) \qquad \forall x \in (-1,1).$$

(a) Determine all values of a for which T_a defines a continuous operator from $L^{2019}((0,1))$ to $L^1((0,1))$, and in these cases determine the norm of the operator.

(b) Determine all values of a for which T_a defines a continuous operator from $L^{2018}((0,1))$ to $L^{2019}((0,1))$, and in these cases determine the norm of the operator.

4. Let us consider the open set

$$\Omega := \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < x^{2019}\right\}.$$

(a) Determine if there exists a positive integer m such that

$$\inf\left\{\int_\Omega \left(|u_x|^{2018} + |u_y|^{2019} + u^m\right) dx : u \in C_c^\infty(\Omega),\ \int_\Omega u(x)^2\,dx \leq 1\right\} = -\infty.$$

(b) Determine if there exists a 1-extender from Ω to $\mathbb{R}^2$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.