Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 25 December 2018

1. Let Ω be the ball in $\mathbb{R}^3$ with center in $(5,4,3)$ and radius 2. Let us consider the problem

$$\min\left\{\int_\Omega \left(x^{25}\,u_x^2 + y^{12}\,u_y^2 + z^{2018}\,u_z^2 + u^{8102}\right)dx\,dy\,dz : u \equiv 1 \text{ on } \partial\Omega\right\}.$$

   (a) Determine the Euler-Lagrange equation for the problem.

   (b) Discuss existence, uniqueness, and regularity of the solution.

2. Let us consider the function $F : \ell^{25} \to \ell^{12}$ defined by

$$F(x_1,\ldots,x_n,\ldots) := (x_1^{2018},\ldots,x_n^{2018},\ldots).$$

   Determine if this function is linear, continuous, surjective, compact.

3. Let us consider the open set $\Omega := (-1,1)^3 \subseteq \mathbb{R}^3$. For every positive real number p, let us consider the problem

$$\inf\left\{\int_\Omega \left(|u_x|^{25} + |u_y|^{12} + |u_z|^{2018} - |u|^p\right) :\ u \in C_c^\infty(\Omega)\right\}.$$

   Determine whether the infimum is a real number in each of the following special cases:

   (a) $p = 11$,

   (b) $p = 2000$,

   (c) $p = 3000$.

4. Let us consider a function $f \in W^{25,12}(\mathbb{R}^{2018})$.

   (a) Determine for which values of p we can conclude that $f_{x_1,x_2,x_3,x_4,x_5} \in L^p(\mathbb{R}^{2018})$.

   (b) Let us set $g(x_1,\ldots,x_{2017}) := f(x_1,\ldots,x_{2017},x_1+\ldots+x_{2017})$.
   Determine for which values of q we can conclude that $g \in L^q(\mathbb{R}^{2017})$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.