Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 18 Settembre 2018

1. Let us consider the functional

$$F(u) = \int_0^2 \left(\dot{u}^2 + (u - x^2)^2 \right) dx.$$

Discuss the minimum problem for $F(u)$ with each of the following sets of boundary conditions:

(a) $u(0) = 0$ and $u(2) = 4$.

(b) $u(0) = 0$ and $u'(2) = 4$.

2. Discuss existence, uniqueness and regularity of solutions to equation

$$\left(\dot{u}^2 + e^{\dot{u}} \right) \cdot \ddot{u} = x^2 + e^u$$

with each of the following sets of boundary conditions:

(a) $u(-1) = u(1) = 0$,

(b) $u'(-1) = u'(1) = 0$.

3. Let us consider, for every $\ell > 0$ and $\mu \in \mathbb{R}$, the problem

$$\min \left\{ \int_0^\ell \left(\dot{u}^2 + \sin u - u^2 \right) dx : u \in C^1([0, \ell]),\ u(0) = 0,\ u(\ell) = \mu \right\}.$$

(a) Discuss the existence of the minimum in the case $\ell = \mu = 2018$.

(b) Discuss the existence of the minimum in the case $\ell = 3$ and $\mu = 0$.

(c) Discuss the existence of the minimum in the case $\ell = 3$ and $\mu = 2018$.

4. Let us set

$$m_\varepsilon := \inf \left\{ \int_0^1 \left(\frac{\dot{u}^6}{\varepsilon^2} - \frac{\dot{u}^2}{\varepsilon^6} + \arctan u \right) dx : u \in C^1([0, 1]),\ u(0) = 0,\ u(1) = 0 \right\}.$$

(a) Determine for which positive values of ε it turns out that m_ε is finite.

(b) Determine for which positive values of ε it turns out that m_ε is actually a minimum.

(c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.