1. Discuss the minimum problem

$$\min\left\{\int_0^\ell (\dot u^2 + u\sin x)\,dx : u(0)=u(\ell)\right\}$$

depending on the parameter $\ell > 0$.

(ELE) $(L_p)' = L_s$: $2\ddot u = \sin x \Rightarrow u(x) = ax+b-\frac{1}{2}\sin x$

(BC) $\begin{cases} u(0)=u(\ell) \\ \dot u(0) = \dot u(\ell) \end{cases} \iff \begin{cases} \cancel{b} = a\ell + \cancel{b} - \frac{1}{2}\sin\ell \\ \cancel{a} - \frac{1}{2} = \cancel{a} - \frac{1}{2}\cos\ell \end{cases} \iff \cos\ell = 1 \iff \ell = 2k\pi$

We distinguish two cases

- $\ell = 2k\pi$ for some integer $k \geq 1 \Rightarrow u_0(x) = b - \frac{1}{2}\sin x$ is a solution to (ELE)+(BC) for every $b \in \mathbb{R}$. They are all GM due to the convexity of the Lagrangian (which is NOT strictly convex).

- $\ell \neq 2k\pi$ for every integer $k \geq 1 \Rightarrow$ (ELE) + (BC) has no solution
$\Rightarrow$ min does not exist.

Actually $\inf = -\infty$ and it is approached by the constant functions $u_n(x) := n$ or $u_n(x) := -n$ depending on the sign of

$$\int_0^\ell \sin x\,dx$$

(which is $\neq 0$ in this case).

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2. Discuss existence, uniqueness, regularity of the solution to the boundary value problem

$$\ddot{u} = \frac{u-3}{3+\cos\dot{u}}, \qquad u(0) = 2017, \quad u'(2017) = 0.$$

$(3+\cos\dot{u})\,\ddot{u} = u-3 \iff (3\dot{u}+\sin\dot{u})' = u-3$

This is (ELE) of $\min\{F(u) : u(0) = 2017\}$ with

$$F(u) := \int_0^{2017} \left(\frac{3}{2}\dot{u}^2 - \cos\dot{u} + \frac{1}{2}u^2 - 3u\right) dx$$

Note that the Lagrangian $L(s,p)$ is convex wrt (s,p).

$-\cos p + \frac{1}{2}s^2 - 3s$ is bounded from below $\Rightarrow$ any bound on $F(u)$ gives a bound on $\|\dot{u}\|_{L^2}$ $\Rightarrow$ due to (BC) gives a bound on $\|u\|_{L^\infty}$
$\Rightarrow$ compactness.

Standard direct method applies $\Rightarrow$ existence (note that the second boundary condition $\dot{u}(2017) = 0$ is the NBC originated by the minimization process).

Convexity of $L(s,p)$ $\Rightarrow$ uniqueness

$L_{pp}(s,p) > 0$ $\Rightarrow$ regularity via bootstrap in the usual way (up to C^∞).

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3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell \left(\sin(\dot u^2) - \sinh(u^2)\right)\,dx : u(0) = u(\ell) = 0\right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(c) Compute the infimum as a function of ℓ.

(a) The quadratic functional associated to the second variation is

$$Q_\ell(v) = \int_0^\ell (\dot v^2 - v^2)\,dx$$

We distinguish three cases.

- $\ell > \sqrt{\pi} \Rightarrow Q_\ell$ is not ≥ 0 (condition (J) fails) $\Rightarrow u_0(x)$ is NOT WLM
- $\ell < \sqrt{\pi} \Rightarrow Q_\ell$ is > 0 $\Rightarrow u_0(x)$ is WLM (it satisfies (E), (L^+), (J^+
- $\ell = \sqrt{\pi} \Rightarrow$ observe that $F(u) < Q_\ell(u)$ if $u \neq u_0$, and therefore

  $F(\varepsilon \sin(\sqrt{\pi} x)) < Q_\ell(\ldots) = 0$

  $\Rightarrow u_0 \equiv 0$ is NOT a WLM.

Conclusion: $\boxed{u_0(x) \equiv 0 \text{ is WLM } \Leftrightarrow \ell < \sqrt{\pi}}$

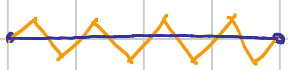
(b) $\boxed{u_0(x) \equiv 0 \text{ is never a SLM}}$ because condition (W) is not satisfied.

Indeed $E(x, u_0(x), \dot u_0(x), q) = L(x, u_0, \dot u_0 + q) - L(x, u_0, \dot u_0) - q L_p(x, u_0, \dot u_0)$

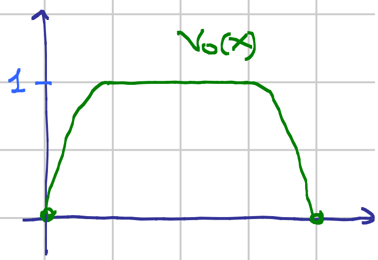
$= \sin(q^2)$

which is not ≥ 0 for every (x, q).

<u>Direct argument</u>: we can approximate $u_0(x) \equiv 0$ in C^0 norm with a sequence u_n with slopes in $\pm\sqrt{\frac{3\pi}{2}}$, so that $F(u_n) \equiv -\ell$.



(c) $\boxed{\inf = -\infty \text{ for every } \ell > 0}$



Take any $v_0(x)$ as in the figure and then observe that

$$F(n v_0(x)) \to -\infty \quad \text{as } n \to +\infty$$

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. Let us set

$$m_\varepsilon = \inf\left\{\int_0^1 \left(\varepsilon \dot{u}^4 - \dot{u}^2 + u^2\right)\, dx : u(0) = u(1) = 2017\right\},$$

where ε is a positive real parameter.

(a) Determine for which values of ε it turns out that $m_\varepsilon \in \mathbb{R}$.

(b) Determine for which values of ε the infimum is actually a minimum.

(c) Determine the leading term of m_ε as $\varepsilon \to 0^+$.

(a) $\boxed{m_\varepsilon \in \mathbb{R} \text{ for every } \varepsilon > 0}$ Indeed, it is enough to remark that $\varepsilon p^4 - p^2 + s^2$ is bounded from below in $\mathbb{R}^2$.

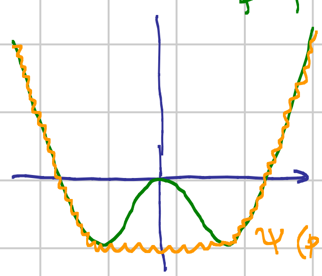
(b) $\boxed{\text{min does NOT exist for every } \varepsilon > 0}$. Assume that $u_0(x)$ is a min. point. Due to BC and Rolle's thm, there exists $x_0 \in (0,1)$ s.t. $\dot{u}(x_0) = 0$, and hence $L_{pp}(x_0, u_0(x_0), \dot{u}_0(x_0)) = -2 < 0$.
$\Longrightarrow$ condition (L) fails $\Longrightarrow$ $u_0(x)$ is not even a DLM.

(c) Let us set $u = \frac{v}{\sqrt{\varepsilon}}$. Then

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\underbrace{\int_0^1 (\dot{v}^4 - \dot{v}^2 + v^2)\, dx}_{G_\varepsilon(v)} \quad \text{with new BC} \quad v(0) = v(1) = 2017\sqrt{\varepsilon}$$

(note: it depends on ε through BC)

Now $G_\varepsilon(v) \xrightarrow{\Gamma} \int_0^1 (\psi(\dot{v}) + v^2)\, dx =: G(v)$ with BC $v(0) = v(1) = 0$, where ψ is the convexification of $p^4 - p^2$.



The verification of Γ-convergence is standard (the only point is keeping BC into account when defining recovery sequences).
Also equicoerciveness follows from inequality $p^4 - p^2 + s^2 \geq p^2 + s^2 - A$ for a suitable constant A.
It follows that

$$\varepsilon m_\varepsilon = \min\left\{G_\varepsilon(v) : v(0) = v(1) = 2017\sqrt{\varepsilon}\right\}$$
$$\to \min\left\{G(v) : v(0) = v(1) = 0\right\} = \psi(0) \text{ (can be explicitly computed)}$$

$\Rightarrow$ $\boxed{\text{the principal part of } m_\varepsilon \text{ is } \frac{\psi(0)}{\varepsilon}}$.

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