25/09/2017

1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\dot{u}^2 + u^2 - x^2 \dot{u}\right)\, dx.$$

Discuss the minimum problem for $F(u)$ subject to each of the following boundary conditions:

(a) $u(1) = 2u(-1)$,

(b) $u(0) = 0$.

$$\delta F(u,v) = \int_{-1}^{1} (2\dot{u} - x^2)\dot{v} + 2uv = 0$$

$$= \left[(2\dot{u} - x^2)v\right]_{-1}^{1} + \int_{-1}^{1} (-2\ddot{u} + 2x + 2u)v = 0$$

$\Rightarrow$ (ELE) is $\ddot{u} = u + x \rightsquigarrow u(x) = a\cosh x + b\sinh x - x$

Boundary term is

$$0 = (2\dot{u}(1) - 1)\underbrace{v(1)}_{2v(-1)} - (2\dot{u}(-1) - 1)v(-1) = v(-1)\left[4\dot{u}(1) - 2 - 2\dot{u}(-1) + 1\right] = 0$$

$\Rightarrow$ B.C.'s are
$$u(1) = 2u(-1)$$
$$4\dot{u}(1) = 2\dot{u}(-1) + 1$$

$\rightsquigarrow$ Solving the system we obtain a unique solution, which is the unique minimum point since the Lagrangian is strictly convex in (s,p).

(b) Let us consider the two pbms

$$\min\left\{\int_{-1}^{0}(\dots) : u(0) = 0\right\} \qquad \min\left\{\int_{0}^{1}(\dots) : u(0) = 0\right\}$$

Both pbm have a unique solution, that satisfies

$$\begin{cases} \ddot{u} = u + x \\ u(0) = 0 \\ 2\dot{u}(-1) - 1 = 0 \end{cases} \qquad \begin{cases} \ddot{u} = u + x \\ u(0) = 0 \\ 2\dot{u}(1) - 1 = 0 \end{cases}$$

Quite surprisingly, in both cases the solution is

$$u(x) = x - \frac{1}{2\cosh(1)}\sinh x$$, and this is the unique minimizer (again by convexity)

2. Let us consider the boundary value problem

$$\ddot{u} = \frac{u + u^3}{1 + \dot{u}^4}, \qquad u(0) = 1, \quad u(1) = \lambda.$$

(a) Discuss existence, uniqueness, and regularity of the solution.

(b) Determine the values of λ for which the solution is convex.

(a) $F(u) := \int_0^1 \left(\frac{1}{2}\dot{u}^2 + \frac{1}{30}\dot{u}^6 + \frac{1}{2}u^2 + \frac{1}{4}u^4 \right) dx$

and the minimum pbm $\quad \min \{ F(u) : u(0) = 1, \ u(1) = \lambda \}$

Existence of minimizers follows from the direct method

Uniqueness follows from the strict convexity in (s,p) of the Lagr.

(ELE) is

$$-\int \left(\dot{u} + \tfrac{1}{5}\dot{u}^5\right)\dot{v} = \int (u + u^3)\, v$$

which implies that $u + u^3$ is the weak derivative of $\dot{u} + \frac{1}{5}\dot{u}^5$. Now

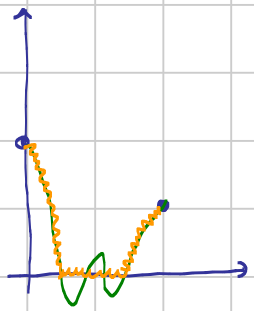
$$u \in H^1 \Rightarrow u \in C^0 \Rightarrow \left(\dot{u} + \tfrac{1}{5}\dot{u}^5\right)' \in C^0 \Rightarrow \dot{u} + \tfrac{1}{5}\dot{u}^5 \in C^1 \Rightarrow \dot{u} \in C^1$$

$x + \frac{1}{5}x^5$ is invertible and inverse function is of class C^1

$\Rightarrow u \in C^2 \Rightarrow \dot{u} + \frac{1}{5}\dot{u}^5 \in C^3 \Rightarrow \ldots$ bootstrap ...

(b) Since $\ddot{u} = \dfrac{u + u^3}{1 + \dot{u}^4}$, we obtain that u convex $\Leftrightarrow u \geq 0$.

So $\lambda \geq 0$ is necessary condition. It is also sufficient because when $\lambda \geq 0$ the minimizer is ≥ 0 due to a truncation argument.



Therefore: the unique minimizer is convex $\Leftrightarrow \lambda \geq 0$.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell \left[(1+u^2)\dot u^2 - 10\sin^2(u) + \cos x \cdot u^6\right] dx : u(0) = u(\ell) = 0\right\}.$$

(a) Determine for which values of ℓ the infimum is actually a minimum.

(b) Determine for which values of ℓ the minimum exists and is negative.

(a) $\boxed{\text{Inf} = \min \iff \ell \le \frac{\pi}{2}}$

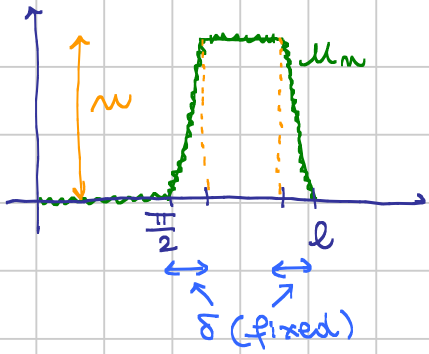
- If $\ell \le \frac{\pi}{2}$, then $\cos x \ge 0$ for every $x \in [0, \ell]$, and therefore we can apply the direct method
  - * CPT : just observe that $F(u) \le M \Rightarrow \int_0^\ell \dot u^2 \le M + 10\ell$
  - * LSC : need some care when passing $\int u_m^2 \dot u_m^2$ to the limit.

$$\int_0^\ell u_m^2 \dot u_m^2 = \int_0^\ell u_\infty^2 \dot u_m^2 + \underbrace{\int_0^\ell \underbrace{(u_m^2 - u_\infty^2)}_{\to 0 \text{ unif}} \underbrace{\dot u_m^2}_{\text{bounded in } L^2}}_{\to 0}$$

$$\int_0^\ell u_\infty^2 \dot u_m^2 = \int_0^\ell u_\infty^2 \left[\dot u_\infty + (\dot u_m - \dot u_\infty)\right]^2 \ge \int_0^\ell u_\infty^2 \dot u_\infty^2 + \underbrace{2\int_0^\ell u_\infty^2 (\dot u_m - \dot u_\infty)}_{\to 0}$$

- If $\ell > \frac{\pi}{2}$, then $\inf = -\infty$

Enough to consider u_m as in figure and obtain that $F(u_m) \sim m^4 - m^6$



(b) $\boxed{\min < 0 \iff \frac{\pi}{\sqrt{10}} < \ell \le \frac{\pi}{2}}$

Let us consider $Q(v) = \frac{1}{2}\delta^2 F(0, v) = \int_0^\ell \dot v^2 - 10 v^2$

- If $\ell > \frac{\pi}{2}$, then (J) fails, and hence $u_0(x) \equiv 0$ is not DLM, and a fortiori it is not GM.

- If $\ell \le \frac{\pi}{2}$, then $F(u) \ge Q(u) \ge 0$.

$(1+u^2)\dot u^2 \ge \dot u^2$

$-10\sin^2 u \ge -10u^2$

$\cos x \cdot u^6 \ge 0$

4. Let us consider, for every $\varepsilon > 0$, the problem

$$m_\varepsilon = \inf\left\{ \int_0^2 \left(\varepsilon \dot{u}^4 - \dot{u}^2 + \varepsilon^2 u^4 \right) dx : u(0) = u(2) = 2 \right\}.$$

(a) Determine for which values $\varepsilon > 0$ the infimum is a real number.

(b) Determine for which values $\varepsilon > 0$ the infimum is actually a minimum.

(c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

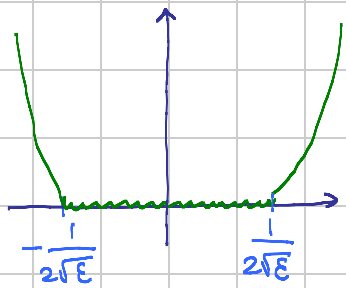
(a) $\boxed{\inf \in \mathbb{R} \quad \forall \varepsilon > 0}$ Enough to observe that $\varepsilon p^4 - p^2$ is bounded from below for every $\varepsilon > 0$.

(b) $\boxed{\inf \neq \min \quad \forall \varepsilon > 0}$ For any $u(x)$ satisfying the BC's there exists $x_0 \in (0,2)$ s.t. $\dot{u}(x_0) = 0$.

In x_0 the second variation does not satisfy (L) (the coeff. of $\dot{v}^2$ is -2).

(c) $\boxed{m_\varepsilon \sim -\frac{1}{2\varepsilon} \quad \text{as } \varepsilon \to 0^+}$ Indeed

$$F(u) = \frac{1}{\varepsilon} \int_0^2 \left[\left(\varepsilon \dot{u}^2 - \frac{1}{2} \right)^2 - \frac{1}{4} + \varepsilon^3 u^4 \right] \geq -\frac{1}{2\varepsilon}$$



and
$$F(u) = -\frac{1}{2\varepsilon} + \frac{1}{\varepsilon} \int_0^2 \left[\psi_\varepsilon(\dot{u}) + \varepsilon^3 u^4 \right]$$

If we consider $u_0(x) \equiv 2$, then

$$m_\varepsilon \leq F(u_0) = -\frac{1}{2\varepsilon} + \varepsilon^2 \int_0^2 16$$

which completes the proof.

— o — o —