Sottospazi vettoriali 4

Argomenti: somma ed intersezione di sottospazi **Difficoltà**: $\star\star\star\star$

Prerequisiti: tutto su sottospazi vettoriali, Span, dimensione, sistemi lineari

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Negli esercizi seguenti vengono assegnati uno spazio vettoriale X e due sottospazi vettoriali V e W, definiti in vario modo. Determinare la dimensione di V, W, $V \cap W$, $V + W$. Determinare anche una base per ciascuno di questi quattro sottospazi.

1. Spazio vettoriale $X = \mathbb{R}^3$.
   (a) $V = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y = z\}$, $\quad W = \operatorname{Span}\{(1, -1, 1), (0, 1, 0)\}$;
   (b) $V = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y = z,\ x - z = 0\}$,
   $W = \{(x, y, z) \in \mathbb{R}^3 : 3x + 2y = z,\ x + z = 0\}$;
   (c) $V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$, $\quad W = \operatorname{Span}\{(1, 2, 3)\}$

2. Spazio vettoriale $X = \mathbb{R}^4$.
   (a) $V = \{(x, y, z, w) \in \mathbb{R}^4 : x + y = z + w,\ x + z = y + w\}$,
   $W = \operatorname{Span}\{(1, 0, 0, -1), (1, 2, 3, 4)\}$;
   (b) $V = \{(x, y, z, w) \in \mathbb{R}^4 : x + y = z + w\}$, $\quad W = \{(x, y, z, w) \in \mathbb{R}^4 : x + z = y + w\}$;
   (c) $V = \{(x, y, z, w) \in \mathbb{R}^4 : x = y,\ z = w\}$,
   $W = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0,\ x - y + z - w = 0\}$.

3. Spazio vettoriale $X = \mathbb{R}_{\leq 4}[x]$.
   (a) $V = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p(x) = p(-x) \quad \forall x \in \mathbb{R}\}$,
   $W = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p(x) = -p(-x) \quad \forall x \in \mathbb{R}\}$;
   (b) $V = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p(1) = 0\}$, $\quad W = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p(-\sqrt{2}) = p(\sqrt{2}) = 0\}$;
   (c) $V = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p(x) = p(-x) \quad \forall x \in \mathbb{R}\}$,
   $W = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p'(0) = p(3) = 0\}$ (si intende che $p'(x)$ è la derivata di $p(x)$).

4. Spazio vettoriale $X = M_{3\times 3}$. Poniamo $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ e $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
   (a) $V = \{A \in M_{3\times 3} : A = A^t\}$, $\quad W = \{A \in M_{3\times 3} : A = -A^t\}$;
   (b) $V = \{A \in M_{3\times 3} : A = A^t\}$, $\quad W = \{A \in M_{3\times 3} : AB = 0\}$;
   (c) $V = \{A \in M_{3\times 3} : AB = BA\}$, $\quad W = \{A \in M_{3\times 3} : Av = 0\}$;
   (d) $V = \{A \in M_{3\times 3} : A = A^t\}$, $\quad W = \operatorname{Span}\{B, B^t\}$;
   (e) $V = \{A \in M_{3\times 3} : BA = 0\}$, $\quad W = \{A \in M_{3\times 3} : Av = 0\}$;
   (f) $V = \{A \in M_{3\times 3} : BA = 0\}$, $\quad W = \{A \in M_{3\times 3} : AB = 0\}$;
   (g) $V = \{A \in M_{3\times 3} : BAv = 0\}$, $\quad W = \{A \in M_{3\times 3} : ABv = 0\}$.

   (Occhio al significato diverso che il simbolo "0" ha nelle varie formule)

1. Spazio vettoriale $X = \mathbb{R}^3$.

(a) $V = \{(x,y,z) \in \mathbb{R}^3 : 2x + 3y = z\}$, $\quad W = \mathrm{Span}\,\{(1,-1,1),(0,1,0)\}$;

(b) $V = \{(x,y,z) \in \mathbb{R}^3 : 2x + 3y = z,\ x - z = 0\}$,
$\quad W = \{(x,y,z) \in \mathbb{R}^3 : 3x + 2y = z,\ x + z = 0\}$;

(c) $V = \{(x,y,z) \in \mathbb{R}^3 : x + y + z = 0\}$, $\quad W = \mathrm{Span}\,\{(1,2,3)\}$

(a) V $\quad$ DIM $V = 2$ $\quad$ BASE: $\{(\overset{v_1}{3,-2,0}),(\overset{v_2}{1,0,2})\}$

W $\quad$ DIM $W = 2$ $\quad$ BASE: $\{(\overset{w_1}{1,-1,1}),(\overset{w_2}{0,1,0})\}$

$V+W$

$$A = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & w_1 & w_2 \end{array} \\ \begin{pmatrix} 3 & 1 & 1 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \end{array} \rightsquigarrow \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 2 & -3 \end{pmatrix}$$

DIM $V+W = 3$

BASE: $\{v_1, v_2, w_1\}$

$V \cap W$ $\quad$ $\dim(V\cap W) = \dim V + \dim W - \dim(V+W) = 2 + 2 - 3 = 1$

$$a v_1 + b v_2 = c w_1 + d w_2 \Rightarrow a v_1 + b v_2 - c w_1 - d w_2 = 0$$

$$\Rightarrow A \cdot \begin{pmatrix} a \\ b \\ -c \\ -d \end{pmatrix} = 0 \Rightarrow \begin{cases} 3a = -b + c \\ 2b = -c + 3d \\ 2c = 3d \end{cases} \quad \begin{cases} 3a = \frac{3}{4}\delta & a = \delta/4 \\ 2b = \frac{3}{2}\delta & b = \frac{3}{4}\delta \\ d = \delta & c = \frac{3}{2}\delta \end{cases}$$

$$\Rightarrow \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 6 \end{pmatrix}$$

BASE: $\{(3,-2,3)\}$

(b) V $\quad \begin{cases} 2x + 3y = z \\ x - z = 0 \end{cases}$ $\quad$ DIM $V = 1$ $\quad$ BASE: $(\overset{v_1}{3,-1,3})$

W $\quad \begin{cases} 3x + 2y = z \\ x + z = 0 \end{cases}$ $\quad$ DIM $W = 1$ $\quad$ BASE: $\{(\overset{w_1}{1,-2,-1})\}$

$V+W$

$$\begin{array}{cc} \begin{pmatrix} 3 & -1 & 3 \\ 1 & -2 & -1 \end{pmatrix} & \begin{array}{c} v_1 \\ w_1 \end{array} \end{array} \rightsquigarrow \begin{pmatrix} 3 & -1 & 3 \\ 0 & 5 & 6 \end{pmatrix}$$

DIM $V+W = 2$

BASE: $\{v_1, w_1\}$

$V \cap W$ $\quad$ $\dim(V\cap W) = 1 + 1 - 2 = 0$

(c) V $\quad x+y+z=0 \quad$ $\dim V = 2 \quad$ BASE: $\{\overset{v_1}{(1,-1,0)}\ \overset{v_2}{(1,0,-1)}\}$

W $\quad \text{SPAN}\{(1,2,3)\} \quad$ $\dim W = 1 \quad$ BASE: $\{\overset{w_1}{(1,2,3)}\}$

$V+W$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{pmatrix}$$

$\dim(V+W) = 3$

BASE: $\{v_1, v_2, w_1\}$

$V \cap W$ $\quad \dim(V\cap W) = 2+1-3 = 0$

2. Spazio vettoriale $X = \mathbb{R}^4$.

(a) $V = \{(x,y,z,w) \in \mathbb{R}^4 : x+y = z+w,\ x+z = y+w\}$,
$W = \text{Span}\{(1,0,0,-1),(1,2,3,4)\}$;

(b) $V = \{(x,y,z,w) \in \mathbb{R}^4 : x+y = z+w\}$, $\quad W = \{(x,y,z,w) \in \mathbb{R}^4 : x+z = y+w\}$;

(c) $V = \{(x,y,z,w) \in \mathbb{R}^4 : x = y,\ z = w\}$,
$W = \{(x,y,z,w) \in \mathbb{R}^4 : x+y+z+w = 0,\ x-y+z-w = 0\}$.

(a) V $\quad \begin{cases} x+y = z+w \\ x+z = y+w \end{cases} \rightsquigarrow \begin{cases} x = w \\ z = y \end{cases}$ $\quad \dim V = 2 \quad$ BASE: $(1,0,0,1)\ v_1$, $(0,1,1,0)\ v_2$

W $\quad \text{SPAN}\{\overset{w_1}{(1,0,0,-1)},\overset{w_2}{(1,2,3,4)}\}$ $\quad \dim W = 2 \quad$ BASE: $\{w_1, w_2\}$

$V+W$

$$\begin{array}{cccc} v_1 & v_2 & w_1 & w_2 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 1 & 1 & -1 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\dim(V+W) = 4 \quad$ BASE: $\{v_1, v_2, w_1, w_2\}$

$V \cap W$ $\quad \dim(V\cap W) = 2+2-4 = 0$

(b) V $\quad x+y = z+w \quad$ $\dim V = 3 \quad$ BASE: $\begin{cases} (1,-1,0,0) & v_1 \\ (1,0,1,0) & v_2 \\ (1,0,0,1) & v_3 \end{cases}$

W $\quad x+z = y+w \quad$ $\dim W = 3 \quad$ BASE: $\begin{cases} (1,1,0,0) & w_1 \\ (1,0,-1,0) & w_2 \\ (1,0,0,1) & w_3 \end{cases}$

$V+W$

$$A=\begin{array}{cccccc} v_1 & v_2 & v_3 & w_1 & w_2 & w_3 \end{array}$$
$$A=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & -2 & -2 & 0 \end{pmatrix}$$

$\dim(V+W)=4$ BASE: $\{v_1, v_2, v_3, w_1\}$

$V\cap W$ $\dim(V\cap W) = 3+3-4 = 2$

$$A\begin{pmatrix} a \\ b \\ c \\ -d \\ -e \\ -f \end{pmatrix} = 0 \Rightarrow \begin{cases} a=-d \\ b=-e \\ -c=-f \\ d=-e \end{cases} \quad \begin{cases} a=-\delta \\ b=-\delta \\ c=s \quad f=s \\ d=\delta \quad e=-\delta \end{cases}$$

$$\begin{cases} \delta=1,\ s=0 \Rightarrow -v_1-v_2 = w_1-w_2 \\ \delta=0 \quad s=1 \Rightarrow v_3 = w_3 \end{cases} \Rightarrow \text{BASE: } \{(v_1+v_2),\ v_3\}$$

(c) V $\begin{cases} x=y \\ z=w \end{cases}$ $\dim V = 2$ SPAN: $\overset{v_1}{(1,1,0,0)}$ $\overset{v_2}{(0,0,1,1)}$

W $\begin{cases} x+y+z+w=0 \\ x-y+z-w=0 \end{cases} \Rightarrow \begin{cases} x+z=0 \\ y+w=0 \end{cases}$ $\dim W = 2$ SPAN: $\overset{w_1}{(1,0,-1,0)}$ $\overset{w_2}{(0,1,0,-1)}$

$V+W$

$$A=\begin{array}{cccc} v_1 & v_2 & w_1 & w_2 \end{array}$$
$$A=\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & \overset{0}{-1} & \overset{0}{1} \end{pmatrix}$$

$\dim(V+W)=3$

SPAN: $\{v_1, v_2, w_1\}$

$V\cap W$

$$A\begin{pmatrix} a \\ b \\ -c \\ -d \end{pmatrix} = 0 \Rightarrow \begin{cases} a=c \\ b=-c \\ c=d \end{cases} \Rightarrow \begin{cases} a=\delta \\ b=-\delta \\ d=\delta \quad c=\delta \end{cases} \Rightarrow \delta(v_1-v_2) = \delta(w_1+w_2)$$

$\dim(V\cap W) = 2+2-3 = 1$ SPAN: $\{(v_1-v_2)\}$

3. Spazio vettoriale $X = \mathbb{R}_{\leq 4}[x]$.

(a) $V = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p(x) = p(-x) \quad \forall x \in \mathbb{R}\}$,
$W = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p(x) = -p(-x) \quad \forall x \in \mathbb{R}\}$;

(b) $V = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p(1) = 0\}$, $\qquad W = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p(-\sqrt{2}) = p(\sqrt{2}) = 0\}$;

(c) $V = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p(x) = p(-x) \quad \forall x \in \mathbb{R}\}$,
$W = \{p(x) \in \mathbb{R}_{\leq 4}[x] : p'(0) = p(3) = 0\}$ (si intende che $p'(x)$ è la derivata di $p(x)$).

(a) V $\quad p(x) = ax^4 + \cancel{b}x^3 + cx^2 + \cancel{d}x + e = ax^4 - \cancel{b}x^3 + cx^2 - \cancel{d}x + e \Rightarrow b = d = 0$

DIM $V = 3$ $\quad$ BASE: $\{x^4, x^2, 1\}$

W $\quad p(x) = \cancel{a}x^4 + bx^3 + \cancel{c}x^2 + dx + \cancel{e} = -\cancel{a}x^4 + bx^3 - \cancel{c}x^2 + dx - \cancel{e} \Rightarrow a = c = e = 0$

DIM $V = 2$ $\quad$ BASE: $\{x^3, x\}$

V+W $\quad$ DIM$(V+W) = 5$ $\quad$ BASE: $\{x^4, x^3, x^2, x, 1\}$

V∩W $\quad$ DIM$(V \cap W) = 3 + 2 - 5 = 0$

(b) V $\quad p(1) = a + b + c + d + e = 0$

DIM $V = 4$ $\quad$ BASE: $\{x^4 - 1, x^3 - 1, x^2 - 1, x - 1\}$

W $\quad p(-\sqrt{2}) = 4a - \cancel{2\sqrt{2}b} + 2c - \cancel{\sqrt{2}d} + e = 4a + \cancel{2\sqrt{2}b} + 2c + \cancel{\sqrt{2}d} + e \quad b = d = 0$

DIM $W = 3$ $\quad$ BASE: $\{x^4, x^2, 1\}$

V+W $\quad$ DIM$(V+W) = 5$ $\quad$ BASE: $\{x^4, x^3 - 1, x^2, x - 1, 1\}$

V∩W $\quad$ DIM$(V \cap W) = 4 + 3 - 5 = 2$ $\quad$ BASE: $\{x^4 - 1, x^2 - 1\}$

(c) V $\quad p(x) = p(-x) \Rightarrow b = d = 0$ $\quad$ DIM $V = 3$ $\quad$ BASE: $\{x^4, x^2, 1\}$

W $\quad p'(0) = d = 0 \quad p(3) = 81a + 27b + 9c + e = 0$

DIM $W = 3$ $\quad$ BASE: $\{x^4 - 81, x^3 - 27, x^2 - 9\}$

V+W $\quad$ DIM$(V+W) = 4$ $\quad$ BASE: $\{x^4, x^3 - 27, x^2, 1\}$

V∩W $\quad$ DIM$(V \cap W) = 3 + 3 - 4 = 2$ $\quad$ BASE: $\{x^4 - 81, x^2 - 9\}$

4. Spazio vettoriale $X = M_{3\times 3}$. Poniamo $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ e $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(a) $V = \{A \in M_{3\times 3} : A = A^t\}$, $W = \{A \in M_{3\times 3} : A = -A^t\}$;

(b) $V = \{A \in M_{3\times 3} : A = A^t\}$, $W = \{A \in M_{3\times 3} : AB = 0\}$;

(c) $V = \{A \in M_{3\times 3} : AB = BA\}$, $W = \{A \in M_{3\times 3} : Av = 0\}$;

(d) $V = \{A \in M_{3\times 3} : A = A^t\}$, $W = \mathrm{Span}\,\{B, B^t\}$;

(e) $V = \{A \in M_{3\times 3} : BA = 0\}$, $W = \{A \in M_{3\times 3} : Av = 0\}$;

(f) $V = \{A \in M_{3\times 3} : BA = 0\}$, $W = \{A \in M_{3\times 3} : AB = 0\}$;

(g) $V = \{A \in M_{3\times 3} : BAv = 0\}$, $W = \{A \in M_{3\times 3} : ABv = 0\}$.

(Occhio al significato diverso che il simbolo "0" ha nelle varie formule)

(a) V $A = A^t$ DIM $V = 6$ BASE: $\left\{ \overset{v_1}{\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}}, \overset{v_4}{\begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}}, \overset{v_5}{\begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}}, \overset{v_6}{\begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}} \right\}$

W $A = -A^t$ DIM $W = 3$ BASE $\left\{ \overset{w_1}{\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&0 \end{pmatrix}}, \overset{w_2}{\begin{pmatrix} 0&0&1\\0&0&0\\-1&0&0 \end{pmatrix}}, \overset{w_3}{\begin{pmatrix} 0&0&0\\0&0&1\\0&-1&0 \end{pmatrix}} \right\}$

V+W DIM$(V+W) = 9$ BASE: $\{e_1, \dots, e_9\}$ $e_1 = v_1$ $e_2 = v_2$ $e_3 = v_3$

$e_4 = \frac{1}{2}(v_4 + w_1)$ $e_5 = \frac{1}{2}(v_5 + w_2)$ $e_6 = \frac{1}{2}(v_6 + w_3)$

$e_7 = \frac{1}{2}(v_4 - w_1)$ $e_8 = \frac{1}{2}(v_5 - w_2)$ $e_9 = \frac{1}{2}(v_6 - w_3)$

$V \cap W$ DIM$(V \cap W) = 6 + 3 - 9 = 0$

(b) V $A = A^t$ DIM $V = 6$ BASE: $\left\{ \overset{v_1}{\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}}, \overset{v_4}{\begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}}, \overset{v_5}{\begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}}, \overset{v_6}{\begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}} \right\}$

W $AB = 0$ $B = \begin{pmatrix} 2&1&0\\0&0&1\\2&1&1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2&1&0\\0&0&1\\0&0&0 \end{pmatrix} \Rightarrow \hat{v}B = 0$ $\hat{v} = (1, 1, -1)$

$=$ BASE KER(B^t)

DIM $W = 3$ BASE: $\left\{ \overset{w_1}{\begin{pmatrix} 1&1&-1\\0&0&0\\0&0&0 \end{pmatrix}}, \overset{w_2}{\begin{pmatrix} 0&0&0\\1&1&-1\\0&0&0 \end{pmatrix}}, \overset{w_3}{\begin{pmatrix} 0&0&0\\0&0&0\\1&1&-1 \end{pmatrix}} \right\}$

V+W $w_1 + w_2 - w_3 = \begin{pmatrix} 1&1&-1\\1&1&-1\\-1&-1&1 \end{pmatrix} = v_1 + v_2 + v_3 + v_4 - v_5 - v_6 = \begin{pmatrix} 1&1&-1\\1&1&-1\\-1&-1&1 \end{pmatrix}$

DIM$(V+W) = 8$ BASE: $\{v_1, \dots, v_6, w_1, w_2\}$

$V \cap W$ DIM$(V \cap W) = 6 + 3 - 8 = 1$ BASE: $\{w_1 + w_2 - w_3\}$

(c) V $AB = BA$ (CASO NON BANALE $\Rightarrow$ SISTEMONE BOVINO!!!)

$$\Rightarrow \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\rightarrow \begin{cases} \cancel{2a} + 2c = \cancel{2a} + d \\ a + c = 2b + e \\ b + c = 2c + f \end{cases} \qquad \begin{cases} 2d + 2f = g \\ d + f = h \\ e + f = i \end{cases} \qquad \begin{matrix} 2g + 2i = 2a + d + g \\ g + i = 2b + e + h \\ \rightarrow h + \cancel{i} = 2c + f + \cancel{i} \end{matrix}$$

$$\begin{cases} d = 2c \\ g = 2h \end{cases} \qquad \begin{cases} a - 2b + c - e = 0 \\ b - c - f = 0 \\ 2c + f - h = 0 \\ e + f - i = 0 \\ 2a + 2c - 2h - 2i = 0 \\ 2b + e - h - i = 0 \end{cases} \qquad \begin{array}{c} \begin{array}{ccccccc} a & b & c & e & f & h & i \end{array} \\ \begin{pmatrix} 1 & -2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 2 & 0 & 2 & 0 & 0 & -2 & -2 \\ 0 & 2 & 0 & 1 & 0 & -1 & -1 \end{pmatrix} \end{array}$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{ccccccc} a & b & c & e & f & h & i \end{array} \\ \begin{pmatrix} 1 & -2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 4 & 0 & 2 & 0 & -2 & -2 \\ 0 & 2 & 0 & 1 & 0 & -1 & -1 \end{pmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{array}{ccccccc} a & b & c & e & f & h & i \end{array} \\ \begin{pmatrix} 1 & -2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 & -1 & -1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc} a & b & c & e & f & h & i \end{array} \\ \begin{pmatrix} 1 & -2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \begin{cases} a = 2b - c + e = \cancel{f} + h + f/2 - h/2 - \cancel{f} + i \\ b = c + f = f/2 + h/2 \qquad \hookrightarrow = f/2 + h/2 + i \\ c = -f/2 + h/2 \Rightarrow d = -f + h \\ e = -f + i \\ g = 2h \end{cases}$$

$$\begin{array}{ll} & \begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\ f = 2, h = 0, i = 0 \Rightarrow & \begin{pmatrix} 1 & 1 & -1 & -2 & -2 & 2 & 0 & 0 & 0 \end{pmatrix} \equiv ? \\ h = 2, f = 0, i = 0 \Rightarrow & \begin{pmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 4 & 2 & 0 \end{pmatrix} \equiv ? \\ i = 1, f = 0, h = 0 \Rightarrow & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \equiv I \end{array}$$

$$\text{DIM } V = 3 \quad \text{BASE: } \left\{ \begin{array}{ccc} V_1 & V_2 & V_3 \equiv I \\ \begin{pmatrix} 1 & 1 & -1 \\ -2 & -2 & +2 \\ 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 5 & 2 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \right\}$$

VERIFICA: $B \cdot V_1 = V_1 \cdot B = 0 \qquad B \cdot V_2 = V_2 \cdot B = \begin{pmatrix} 5 & 2 & 2 \\ 5 & 2 & 0 \\ 8 & 5 & 2 \end{pmatrix} \qquad B \cdot V_3 = V_3 \cdot B = B$

OSS: $\begin{cases} V_1 + V_2 + 2V_3 = \begin{pmatrix} 5 & 2 & 0 \\ 0 & 0 & 2 \\ 5 & 2 & 2 \end{pmatrix} = 2B \\ \text{IN GENERALE } B^m = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 \qquad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \qquad m \in \mathbb{N} \end{cases}$

$W \qquad Av = 0 \quad v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow A = \begin{pmatrix} a & 0 & d \\ b & 0 & e \\ c & 0 & f \end{pmatrix}$

$$\text{DIM } W = 6 \quad \text{BASE: } \left\{ \begin{array}{cccccc} W_1 & W_2 & W_3 & W_4 & W_5 & W_6 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \right\}$$

$V + W \qquad \text{DIM}(V + W) = 9 \qquad \text{BASE: } \{ V_1, V_2, V_3, W_1, \ldots, W_6 \}$

$V \cap W \qquad \text{DIM}(V \cap W) = 0$

(d) V $\quad A = A^T \quad$ DIM V = 6 BASE:

$$\left\{ \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, & \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}, & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}, & \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, & \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}, & \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix} \end{array} \right\}$$

W $\quad$ SPAN$\{B, B^T\}$

DIM W=2 $\quad$ BASE: $\{B, B^T\}$

$$B = \begin{pmatrix} 2&1&0\\0&0&1\\2&1&1 \end{pmatrix} \qquad B^T = \begin{pmatrix} 2&0&2\\1&0&1\\0&1&1 \end{pmatrix}$$

V+W $\quad B + B^T = \begin{pmatrix} 2&1&2\\1&0&2\\2&2&1 \end{pmatrix} = 2v_1 + v_3 + v_4 + 2v_5 + 2v_6$

DIM(V+W) = 7 $\qquad$ BASE: $\{v_1, \dots, v_6, B\}$

V∩W $\quad$ DIM(V∩W) = 6 + 2 − 7 = 1 $\quad$ BASE: $\{B + B^T\}$

(e) V

$$BA = 0 \quad B = \begin{pmatrix} 2&1&0\\0&0&1\\2&1&1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2&1&0\\0&0&1\\0&0&0 \end{pmatrix} \quad B\hat{v} = 0 \Rightarrow \hat{v} = \begin{pmatrix} 1\\-2\\0 \end{pmatrix} = \text{BASE KER } B$$

DIM(V) = 3 $\quad$ BASE:

$$\left\{ \begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{pmatrix} 1&0&0\\-2&0&0\\0&0&0 \end{pmatrix}, & \begin{pmatrix} 0&1&0\\0&-2&0\\0&0&0 \end{pmatrix}, & \begin{pmatrix} 0&0&1\\0&0&-2\\0&0&0 \end{pmatrix} \end{array} \right\}$$

W

$$Av = 0 \quad v = \begin{pmatrix} 0\\1\\0 \end{pmatrix} \Rightarrow A = \begin{pmatrix} a&0&d\\b&0&e\\c&0&f \end{pmatrix}$$

DIM W = 6 $\quad$ BASE:

$$\left\{ \begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, & \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}, & \begin{pmatrix} 0&0&0\\0&0&0\\1&0&0 \end{pmatrix}, & \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix}, & \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix} \end{array} \right\}$$

$V+W \qquad v_1 = w_1 - 2w_2 \qquad v_3 = w_4 - 2w_5$

$\dim(V+W) = 7 \qquad \text{BASE: } \{v_2, w_1, \dots, w_6\}$

$V \cap W \qquad \dim(V\cap W) = 6+3-7 = 2 \qquad \text{BASE: } \{v_1, v_3\}$

(f) V

$$BA = 0 \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad B\hat{v} = 0 \Rightarrow \hat{v} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \equiv \text{BASE KER } B$$

$$\dim(V) = 3 \quad \text{BASE: } \left\{ \overset{v_1}{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}} \right\}$$

W

$$AB = 0 \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \hat{v}B = 0 \quad \hat{v} = (1, 1, -1) \equiv \text{BASE KER}(B^T)$$

$$\dim W = 3 \quad \text{BASE: } \left\{ \overset{w_1}{\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{w_2}{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{w_3}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}} \right\}$$

$V+W \quad \dim(V+W) = 6 \quad \text{BASE: } \{v_1, v_2, v_3, w_1, w_2, w_3\}$

$V \cap W \quad \dim(V\cap W) = 3+3-6 = 0$

(g) V

$$BAv = 0,\ v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Leftrightarrow Av \in \text{KER } B = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \Rightarrow A = \begin{pmatrix} a & b & d \\ b & -2b & e \\ c & 0 & f \end{pmatrix} \quad \dim V = 7$$

$$\text{BASE: } \left\{ \overset{v_1}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}, \overset{v_4}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{v_5}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{v_6}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \overset{v_7}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \right\}$$

W

$$ABv = 0,\ v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow A \begin{pmatrix} a & d & -a \\ b & e & -b \\ c & f & -c \end{pmatrix} \quad \dim W = 6$$

$$\text{BASE: } \left\{ \overset{w_1}{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{w_2}{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{w_3}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}}, \overset{w_4}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{w_5}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \overset{w_6}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \right\}$$

$V+W$ $\quad w_1 = v_1 - v_4 \quad w_2 = v_2 - v_5 \quad w_3 = v_3 - v_6 \quad v_7 = w_4 - 2w_5$

$\dim(V+W) = 7+6-4 = 9 \quad$ BASE: $\{v_1, \dots, v_6, w_4, w_5, w_6\} \equiv \{e_1, \dots, e_9\}$

$V \cap W$ $\quad \dim(V \cap W) = 7+6-9 = 4$

BASE $\{w_1, w_2, w_3, v_7\}$