Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 29 Giugno 2018

1. Let us consider the two functionals

$$F_1(u) = \int_0^2 \left\{ (\dot{u} - 2x)^2 + (u - x^2)^2 \right\} dx, \qquad F_2(u) = \int_0^2 \left\{ (\dot{u} - 2x) + (u - x^2)^2 \right\} dx.$$

   (a) Discuss the minimum problem for $F_1(u)$ subject to the boundary condition $u(0) = u(2)$.

   (b) Discuss the minimum problem for $F_2(u)$ subject to the boundary condition $u(0) = u(2)$.

2. Let us consider the boundary value problem

$$\ddot{u} = \frac{x^7 u^9}{\dot{u}^8 + 10}, \qquad u'(0) = 11, \quad u(11) = 0.$$

   Discuss existence, uniqueness and regularity of the solution.

3. Let us consider, for every $\lambda > 0$, the problem

$$m_\lambda := \inf \left\{ \int_0^4 \left(\dot{u}^2 - \lambda u \arctan u + \sin^4 u \right) dx : u \in C^1([0,4]),\ u(0) = u(4) = 0 \right\}.$$

   (a) Determine for which values of λ the infimum is actually a minimum.

   (b) Determine for which values of λ the infimum is negative.

   (c) Determine the leading term of m_λ as $\lambda \to +\infty$.

4. Let us consider, for every value of the real parameter a, the minimum problem

$$\min \left\{ \int_0^1 u'(x) \cdot e^{u'(x)}\, dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = a \right\}.$$

   (a) Determine for which values of a the function $u(x) = ax$ is a weak local minimum.

   (b) Determine for which values of a the function $u(x) = ax$ is a strong local minimum.

   (c) Determine for which values of a the minimum exists.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.