$f(x, y, z) = x^2 + y^2 + z^2 - xyz$

Consideriamo poi i due insiemi

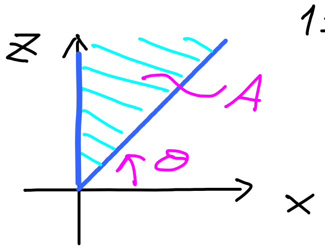
$A = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq y \leq 2,\ 0 \leq x \leq z\}$

$B = \{(x, y, z) \in \mathbb{R}^3 : 1 < y < 2,\ 0 < x < z\}$

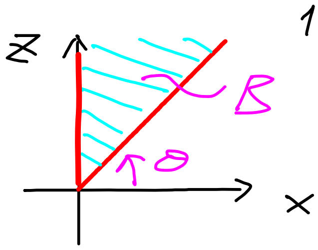
Le domande sono due.

1. (Abbastanza facile) Stabilire se esiste il limite all'infinito di f ristretta ai due insiemi A e B.
2. (Decisamente più delicata) In caso di risposta negativa alla domanda precedente, determinare liminf e limsup (sempre sia per la restrizione ad A sia per la restrizione a B).

$1 \leq y \leq 2$



$1 < y < 2$



COORDINATE CILINDRICHE

$$\begin{cases} x = \rho\cos\theta \\ z = \rho\sin\theta \end{cases} \qquad \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \qquad \rho \to +\infty$$

$$\rightsquigarrow f(x,y,z) = x^2 + y^2 + z^2 - xyz =$$

$$= \rho^2\cos^2\theta + y^2 + \rho^2\sin^2\theta - \rho^2\sin\theta\cos\theta \cdot y =$$

$$= \rho^2(1 - y\sin\theta\cos\theta) + y^2 =$$

$$= \rho^2\left(1 - \frac{1}{2}y\sin 2\theta\right) + y^2$$

1) ESISTENZA

SU A

$$\begin{cases} \theta = \frac{\pi}{4}\ (x = z),\ y = 2 \rightsquigarrow f(x,y,z) = 4 \\ \theta = \frac{\pi}{2}\ (x = 0) \rightsquigarrow f(x,y,z) = \rho^2 + y^2 \to +\infty \end{cases}$$

$\rightsquigarrow$ IL LIMITE NON ESISTE IN A

su B

$$f\left(\sigma, 2-\frac{1}{\sigma}, \sigma+\frac{1}{\sigma}\right) = \sigma^2 + \left(2-\frac{1}{\sigma}\right)^2 + \left(\sigma+\frac{1}{\sigma}\right)^2 +$$

$$- \sigma\left(2-\frac{1}{\sigma}\right)\left(\sigma+\frac{1}{\sigma}\right) = \sigma^2 + 4 + \frac{1}{\sigma^2} - \frac{4}{\sigma} + \sigma^2 + \frac{1}{\sigma^2} + 2 +$$

$$- (2\sigma - 1)\left(\sigma+\frac{1}{\sigma}\right) = \cancel{2\sigma^2} + 6 + \frac{2}{\sigma^2} - \frac{4}{\sigma} - \cancel{2\sigma^2} - 2 + \sigma + \frac{1}{\sigma} =$$

$$= 4 + \frac{2}{\sigma^2} - \frac{3}{\sigma} + \sigma \longrightarrow +\infty$$

$$f\left(\sigma, 2-\frac{1}{\sigma^2}, \sigma+\frac{1}{\sigma}\right) = \sigma^2 + \left(2-\frac{1}{\sigma^2}\right)^2 + \left(\sigma+\frac{1}{\sigma}\right)^2 +$$

$$- \sigma\left(2-\frac{1}{\sigma^2}\right)\left(\sigma+\frac{1}{\sigma}\right) = \sigma^2 + 4 + \frac{1}{\sigma^4} - \frac{4}{\sigma^2} + \sigma^2 + \frac{1}{\sigma^2} + 2 +$$

$$- \left(2\sigma - \frac{1}{\sigma}\right)\left(\sigma+\frac{1}{\sigma}\right) = \cancel{2\sigma^2} + 6 - \frac{3}{\sigma^2} + \frac{1}{\sigma^4} - \cancel{2\sigma^2} - 2 + 1 + \frac{1}{\sigma^2} =$$

$$= 5 - \frac{2}{\sigma^2} + \frac{1}{\sigma^4} \longrightarrow 5$$

$\rightsquigarrow$ IL LIMITE NON ESISTE IN B

2) ERRANZA

SU A

$$\theta = \frac{\pi}{2} \; (x=0) \rightsquigarrow f(x,y,z) = \rho^2 + y^2 \to +\infty$$

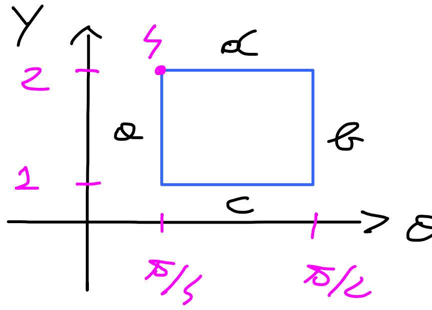
$\rightsquigarrow$ LIMSUP $= +\infty$

$$f(x,y,z) = x^2+y^2+z^2-xyz =$$
$$= \rho^2 \left(1 - \frac{1}{2} y \sin 2\theta\right) + y^2$$

$$\begin{cases} f_x = 2x - yz = 0 \\ f_y = 2y - xz = 0 \\ f_z = 2z - xy = 0 \end{cases} \qquad \begin{cases} x = \dfrac{yz}{2} & \rightsquigarrow x = y = 2 \\ y = \dfrac{xz}{2} = \dfrac{yz^2}{4} & \rightsquigarrow z = 2 \\ x^2 = 4 & \rightsquigarrow x = 2 \end{cases}$$

$\rightsquigarrow$ NON CI SONO PUNTI CRITICI

SIA $\rho \to +\infty$ FISSATO, CERCHIAMO MIN, MAX SUL BORDO



ρ FISSATO

a) $\theta = \frac{\pi}{4}$

$$g(y) = \rho^2 \left(1 - \frac{y}{2}\right) + y^2$$

$$g'(y) = -\frac{1}{2}\rho^2 + 2y = 0 \qquad y = \frac{\rho^2}{4}$$

$\rightsquigarrow$ IMPOSS. APPENA $\rho^2 > 8$

$$g(1) = \frac{1}{2}\rho^2 + 1 > 5 \text{ PER } \rho^2 > 8, \qquad g(2) = 4$$

b) $\theta = \frac{\pi}{2}$

$g(y) = \rho^2 + y^2 \qquad g'(y) = 2y = 0 \qquad y = 0$

$g(1) = \rho^2 + 1 < g(2) = \rho^2 + 4$

c) $y = 1$

$g(\theta) = \rho^2 \left(1 - \frac{1}{2} \sin 2\theta\right) + 1$

$g'(\theta) = -\rho^2 \cos 2\theta = 0 \qquad \theta = \frac{\pi}{4}$

$g(\pi/4) = \rho^2/2 + 1 < g(\pi/2) = \rho^2 + 1$

d) $y = 2$

$g(\theta) = \rho^2 (1 - \sin 2\theta) + 4$

$g'(\theta) = -2\rho^2 \cos 2\theta = 0 \qquad \theta = \frac{\pi}{4}$

$g(\pi/4) = 4 < g(\pi/2) = \rho^2 + 4$

$\rightsquigarrow \forall \rho \quad f(x, y, z) \geq 4$

$\theta = \frac{\pi}{4} \ (x = z), \ y = 2 \quad \rightsquigarrow f(x, y, z) = 4$

$\rightsquigarrow$ LIMINF $= 4$

SIA $\rho \to +\infty$ FISSATO

$$\begin{cases} f_y = -\frac{1}{2}\rho^2 \sin 2\theta + 2y = 0 \rightsquigarrow -\frac{1}{2}\rho^2 + 2y = 0 \\ f_\theta = -\rho^2 y \cos 2\theta = 0 \rightsquigarrow \theta = \frac{\pi}{4} \end{cases}$$

$\rightsquigarrow$ NON ESISTONO PUNTI CRITICI $\rho^2 > 8$

SU B

$$f\left(\sigma, 2 - \frac{1}{\sigma}, \sigma + \frac{1}{\sigma}\right) \to +\infty$$

$\rightsquigarrow$ LIMSUP $= +\infty$

$\rightsquigarrow$ NON CI SONO PUNTI CRITICI

$\rightsquigarrow$ INF$(f) = 4 \quad \forall \rho$

$\rightsquigarrow$ LIMINF $= 4$

$$f\left(\sigma, 2 - \frac{1}{\sigma^3}, \sigma + \frac{1}{\sigma^2}\right) =$$

$$= \cancel{\sigma^2} + 4 + \frac{1}{\sigma^6} - \frac{4}{\sigma^3} + \cancel{\sigma^2} + \frac{1}{\sigma^5} + \cancel{\frac{2}{\sigma}} - \cancel{2\sigma^2} - \cancel{\frac{2}{\sigma}} + \frac{1}{\sigma} + \frac{1}{\sigma^5} \to 4$$