BY GIMUSI 16/10/17
REV.01

Limiti 2

Argomenti: limiti di funzioni di più variabili **Difficoltà**: ⋆⋆⋆⋆

Prerequisiti: tecniche per il calcolo di limiti in un punto per funzioni di più variabili

In ogni riga è assegnata una funzione, di cui si chiede di calcolare liminf e limsup per $(x, y) \to (0, 0)$. Nelle varie colonne, la funzione si intende definita nel suo "naturale dominio" intersecato l'insieme definito dalle relazioni indicate in testa alla colonna stessa.

		a) $(x,y) \in \mathbb{R}^2$		b) $x>0,\ y>0$		c) $0 \le x \le y$		d) $x>0,\ y \le x^2$	
	Funzione	liminf	limsup	liminf	limsup	liminf	limsup	liminf	limsup
1)	$\frac{x^2y}{x^4+y^2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
2)	$\frac{xy^2}{x^4+y^2}$	0	0	0	0	0	0	0	0
3)	$\frac{x^3y}{x^4+y^2}$	0	0	0	0	0	0	0	0
4)	$\frac{xy}{\lvert x\rvert+y^2}$	0	0	0	0	0	0	0	0
5)	$\frac{x}{x^3+y^3}$	$-\infty$	$+\infty$	0	$+\infty$	0	$+\infty$	$-\infty$	$+\infty$
6)	$\frac{xy}{x^3+y^3}$	$-\infty$	$+\infty$	0	$+\infty$	0	$+\infty$	$-\infty$	$+\infty$
7)	$\frac{x^2y}{x^3+y^3}$	$-\infty$	$+\infty$	0	$\frac{2}{3}\sqrt[3]{1/2}$	0	$\frac{1}{2}$	$-\infty$	$+\infty$
8)	$\frac{y^4}{\lvert x\rvert^3+\lvert y\rvert^3}$	0	0	0	0	0	0	0	0
9)	$\frac{y^4}{x^3+y^3}$	$-\infty$	$+\infty$	0	0	0	0	$-\infty$	$+\infty$
10)	$\frac{x+2y}{x+y}$	$-\infty$	$+\infty$	1	2	$\frac{3}{2}$	2	$-\infty$	$+\infty$
11)	$\frac{x^3}{x-y^2}$	$-\infty$	$+\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$
12)	$\frac{\sqrt{x^2+\lvert y\rvert}}{\lvert x\rvert+\lvert y\rvert}$	1	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	1	$+\infty$

 Test di allenamento n. 2

1) $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^5+y^2}$ $\qquad x^2=u\geq 0 \quad y=v$

$$f(x,y)=g(u,v)=\frac{uv}{u^2+v^2}=\frac{1}{2}\sin 2\theta$$

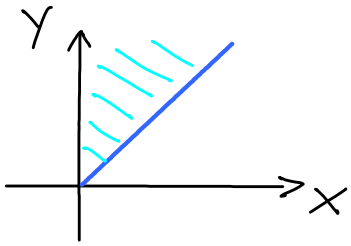
a) $(x,y)\in\mathbb{R}^2$, $u\geq 0$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

LIMINF $=-\frac{1}{2}$ $\qquad$ LIMSUP $=\frac{1}{2}$

b) $x>0, y>0$ $\qquad 0<\theta<\frac{\pi}{2}$

LIMINF $=0$ $\qquad$ LIMSUP $=\frac{1}{2}$

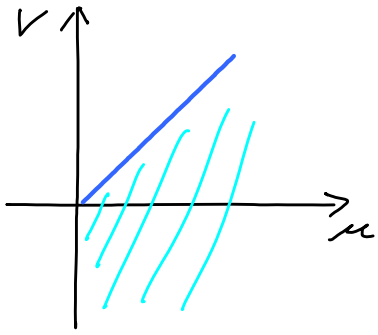
c) $0\leq x\leq y$



$$0\leq\frac{x^2y}{x^5+y^2}\leq\frac{x^2y}{x^5+x^2}=\frac{y}{x^2+1}\to 0$$

LIMINF $=0$ $\qquad$ LIMSUP $=0$

d) $x>0 \quad y\leq x^2$, $u>0 \quad v\leq u$



$$-\frac{1}{2}\leq\frac{uv}{u^2+v^2}\leq\frac{1}{2}$$

LIMINF $=-\frac{1}{2}$ $\qquad$ LIMSUP $=\frac{1}{2}$

2) $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^5+y^2}$

$y^2=v \quad x=u \qquad \dfrac{uv}{u^2+v}=\dfrac{\rho\sin\theta\cos\theta}{\rho\cos^2\theta+\sin\theta}$

$$0\leq\frac{|x|y^2}{x^5+y^2}\leq|x|\to 0 \qquad \text{LIMINF}=\text{LIMSUP}=0$$

3) $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$

$$0 \le \left|\frac{x^3 y}{x^6+y^2}\right| = |x| \left|\frac{x^2 y}{x^6+y^2}\right| \le |x| \frac{1}{2} \to 0$$

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LIMINF = LIMSUP = 0

4) $\lim_{(x,y)\to(0,0)} \frac{xy}{|x|+y^2}$

$x = u^2 \quad y = v \qquad f(x,y) = g(u,v) = \frac{\pm u^2 v}{u^2+v^2}$

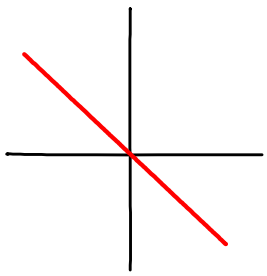
$$0 \le \left|\frac{\pm u^2 v}{u^2+v^2}\right| \le |u| \frac{|uv|}{u^2+v^2} \le \frac{1}{2}|u| \to 0$$

LIMINF = LIMSUP = 0

5) $\lim_{(x,y)\to(0,0)} \frac{x}{x^3+y^3}$ $\left(= \frac{\cos\theta}{\rho^2(\cos^3\theta+\sin^3\theta)} \right)$

a) $(x,y) \in \mathbb{R}^2 \qquad x \pm -y$

$$f(\delta,\delta) = \frac{\delta}{2\delta^3} = \frac{1}{2\delta^2} \to +\infty$$



$$f(\delta^3,\delta^2) = \frac{\delta^3}{\delta^9+\delta^6} = \frac{1}{\delta^6+\delta^3} \to -\infty \qquad \delta \to 0^-$$

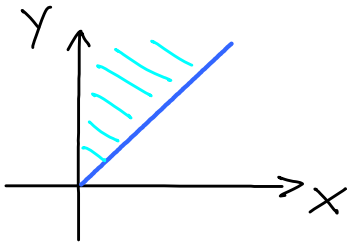
LIMINF = $-\infty$ LIMSUP = $+\infty$

b) $x>0, y>0$ $\quad \dfrac{x}{x^3+y^3} \geq 0$

$f(\delta^5, \delta) = \dfrac{\delta^5}{\delta^{12}+\delta^3} = \dfrac{\delta}{\delta^9+1} \to 0 \quad \rightsquigarrow$ LIMINF = 0

$f(\delta, \delta) = \dfrac{1}{2\delta^2} \to +\infty \quad \rightsquigarrow$ LIMSUP = $+\infty$

c) $0 \leq x \leq y$

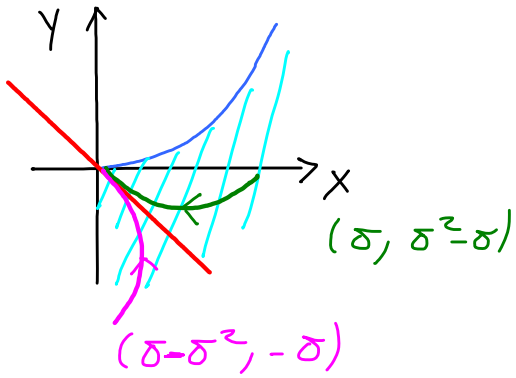


$\dfrac{x}{x^3+y^3} \geq 0$

$f(0, \delta) = 0 \quad \rightsquigarrow$ LIMINF = 0

$f(\delta, \delta) = \dfrac{1}{2\delta^2} \to +\infty \quad \rightsquigarrow$ LIMSUP = $+\infty$

d) $x>0 \quad y \leq x^2$



$$f(\delta, \delta^2-\delta) = \frac{\delta}{\delta^3+\delta^6-3\delta^5+3\delta^4-\delta^3} = \frac{1}{\delta^5-3\delta^4+3\delta^3} \to +\infty \quad \delta \to 0^+$$

$$f(\delta-\delta^2, -\delta) = \frac{\delta-\delta^2}{\delta^3-3\delta^4+3\delta^5-\delta^6-\delta^3} = \frac{1-\delta}{-3\delta^3+3\delta^4-\delta^5} \to -\infty \quad \delta \to 0^+$$

LIMINF = $-\infty$ $\quad$ LIMSUP = $+\infty$

6) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^3+y^3}$

a) $(x,y) \in \mathbb{R}^2 \quad x \neq -y$

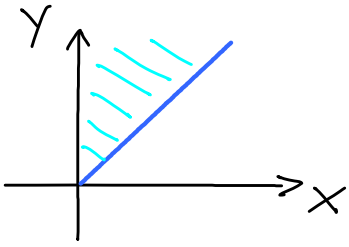
$f(\delta,\delta) = \frac{\delta^2}{2\delta^3} = \frac{1}{2\delta} \to \pm\infty \rightsquigarrow \begin{cases} LIMSUP = +\infty \\ LIMINF = -\infty \end{cases}$

b) $x>0, y>0 \qquad \frac{xy}{x^3+y^3} \geq 0$

$f(\delta^5,\delta) = \frac{\delta^5}{\delta^{12}+\delta^3} = \frac{\delta^2}{\delta^9+1} \to 0 \rightsquigarrow LIMINF = 0$

$f(\delta,\delta) = \frac{1}{2\delta^2} \to +\infty \rightsquigarrow LIMSUP = +\infty$

c) $0 \leq x \leq y$

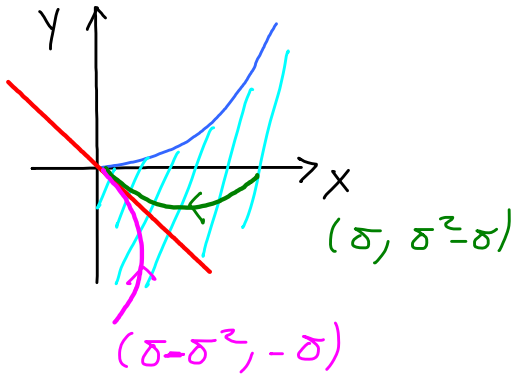


$\frac{x}{x^3+y^3} \geq 0$

$f(0,\delta) = 0 \rightsquigarrow LIMINF = 0$

$f(\delta,\delta) = \frac{1}{2\delta^2} \to +\infty \rightsquigarrow LIMSUP = +\infty$

d) $x>0 \quad y \leq x^2$



$f(\delta, \delta^2-\delta) = \frac{\delta^3-\delta^2}{\delta^3+\delta^6-3\delta^5+3\delta^4-\delta^3} = \frac{\delta-1}{\delta^4-3\delta^3+3\delta^2} \to -\infty \quad \delta \to 0^+$

$f(\delta-\delta^2, -\delta) = \frac{\delta^3-\delta^2}{\delta^3-3\delta^4+3\delta^5-\delta^6-\delta^3} = \frac{\delta-1}{-3\delta^2+3\delta^3-\delta^4} \to +\infty \quad \delta \to 0^+$

$LIMINF = -\infty \qquad LIMSUP = +\infty$

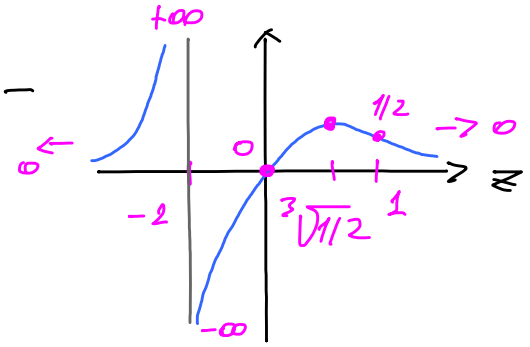
7) $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^3+y^3}$ $\left(= \frac{\cos^2\vartheta \sin\vartheta}{\cos^3\vartheta + \sin^3\vartheta} = \frac{\operatorname{tg}\vartheta}{1+\operatorname{tg}^3\vartheta}, \cos\vartheta \neq 0 \right)$

$\operatorname{tg}\vartheta \neq -1$

$z = \operatorname{tg}\vartheta \in (-\infty, +\infty)$

$g(z) = \frac{z}{1+z^3}$ $\quad g'(z) = \frac{1+z^3-3z^3}{(1+z^3)^2} = \frac{1-2z^3}{(1+z^3)^2}$

$g'(z)$ $\quad + + + \bullet + + + \bullet - - - -$ $\quad -1 \quad \sqrt[3]{1/2}$



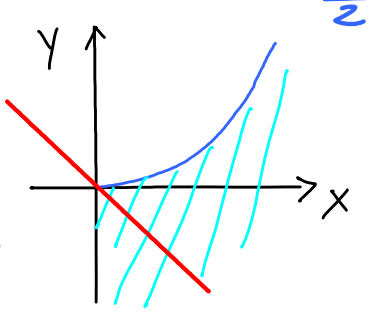
$g(\sqrt[3]{1/2}) = \frac{\sqrt[3]{1/2}}{1+\frac{1}{2}} = \frac{2}{3}\sqrt[3]{1/2}$

a) $(x,y) \in \mathbb{R}^2$ $\quad x \neq -y$ $\quad$ LIMINF $= -\infty$ $\quad$ LIMSUP $= +\infty$

b) $x>0, y>0$ $\quad$ LIMINF $= 0$ $\quad$ LIMSUP $= \frac{2}{3}\sqrt[3]{1/2}$

c) $0 \le x \le y$ $\quad$ LIMINF $= 0$ $\quad$ LIMSUP $= \frac{1}{2}$

d) $x>0$ $\quad y \le x^2$



LIMINF $= -\infty$ $\quad$ LIMSUP $= +\infty$

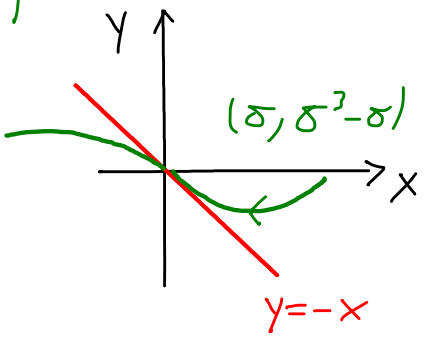
8) $\lim_{(x,y)\to(0,0)} \frac{y^4}{|x|^3+|y|^3}$

$0 \le \frac{y^4}{|x|^3+|y|^3} = \frac{\rho \cos^4\vartheta}{|\cos\vartheta|^3+|\sin\vartheta|^3} \le \frac{\rho \cdot 1}{m} \to 0$

$\ge m > 0$

LIMINF $=$ LIMSUP $= 0$

9) $\lim_{(x,y)\to(0,0)} \frac{y^4}{x^3+y^3}$ $\left(= \frac{\rho \cos^4\theta}{\cos\theta^3 + \sin\theta^3} \right)$



a) $(x,y) \in \mathbb{R}^2 \quad x \neq -y$

$$f(\delta, \delta^3-\delta) = \frac{\delta^4 - 4\delta^6 + 6\delta^8 - 4\delta^{10} + \delta^{12}}{\delta^3 + \delta^9 - 3\delta^7 + 3\delta^5 - \delta^3} =$$

$$= \frac{1 - 4\delta^2 + 6\delta^4 - 4\delta^6 + \delta^8}{\delta^5 - 3\delta^3 + 3\delta} \to \begin{cases} +\infty & \delta \to 0^+ \\ -\infty & \delta \to 0^- \end{cases}$$

LIMINF $= -\infty$ LIMSUP $= +\infty$

b) $x>0, y>0$

$$0 \le \frac{y^4}{x^3+y^3} = \frac{\rho \cos^4\theta}{\cos^3\theta + \sin^3\theta} \le \frac{\rho \cdot 1}{m} \to 0$$

$\geq m > 0$

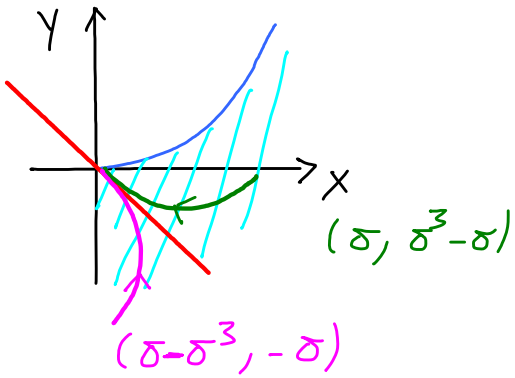
LIMINF $=$ LIMSUP $= 0$

c) $0 \le x \le y$ $0 \le \frac{y^4}{x^3+y^3} = \frac{\rho \cos^4\theta}{\cos^3\theta + \sin^3\theta} \le \frac{\rho \cdot \sqrt{2}/2}{m} \to 0$

$\geq m > 0$

LIMINF $=$ LIMSUP $= 0$

d) $x>0 \quad y \le x^2$



$f(\delta, \delta^3-\delta) \overset{a)}{\to} +\infty \quad \delta \to 0^+$

$$f(\delta-\delta^3, -\delta) = \frac{\delta^4}{\delta^3 - 3\delta^5 + 3\delta^7 - \delta^9 - \delta^3} = \frac{1}{-3\delta + 3\delta^3 - \delta^5} \to -\infty \quad \delta \to 0^+$$

LIMINF $= -\infty$ LIMSUP $= +\infty$

10) $\lim_{(x,y)\to(0,0)} \frac{x+2y}{x+y}$ $\left(= 1 + \frac{y}{x+y} = 1 + \frac{\sin\theta}{\cos\theta+\sin\theta} = 1 + \frac{\operatorname{tg}\theta}{1+\operatorname{tg}\theta} \right)$

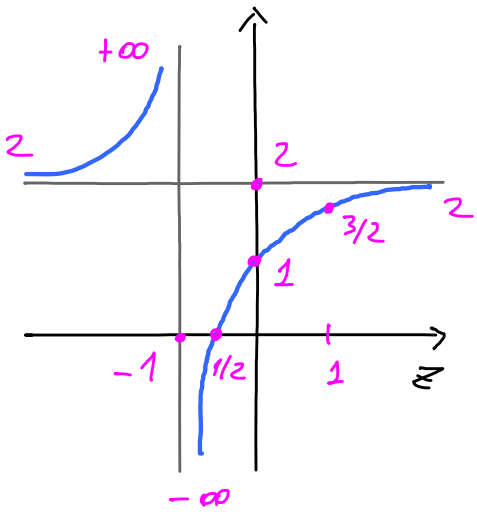
$\cos\theta \neq 0 \quad \operatorname{tg}\theta \neq -1$

$z = \operatorname{tg}\theta \in (-\infty, +\infty)$

$g(z) = 1 + \frac{z}{1+z} \qquad g'(z) = \frac{1+z-z}{(1+z)^2} = \frac{1}{(1+z)^2} > 0$

a) $(x,y) \in \mathbb{R}^2 \qquad x \neq -y$

LIMINF $= -\infty$ LIMSUP $= +\infty$



b) $x > 0, y > 0$

LIMINF $= 1$ LIMSUP $= 2$

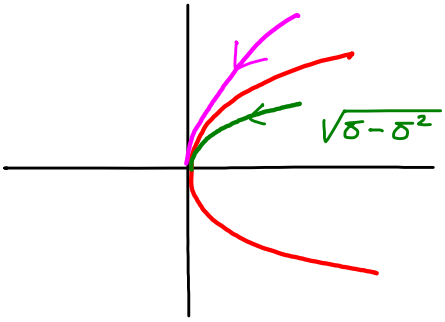
c) $0 \leq x \leq y$

LIMINF $= \frac{3}{2}$ LIMSUP $= 2$

d) $x > 0 \quad y \leq x^2$

LIMINF $= -\infty$ LIMSUP $= +\infty$

11) $\lim_{(x,y)\to(0,0)} \frac{x^3}{x-y^2}$ $\left(= \frac{\rho^2 \cos^3\theta}{\cos\theta - \rho \sin^2\theta} \right)$



a) $(x,y) \in \mathbb{R}^2 \quad x \pm y^2$

$$f(\delta, \sqrt{\delta-\delta^2}) = \frac{\delta}{\delta-\delta+\delta^2} = \frac{1}{\delta} \to +\infty$$

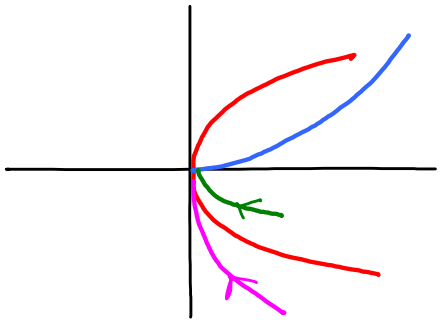
$$f(\delta, \sqrt{\delta+\delta^2}) = \frac{\delta}{\delta-\delta-\delta^2} = -\frac{1}{\delta} \to -\infty$$

LIMINF $= -\infty$ LIMSUP $= +\infty$

b) $x>0, y>0$ $\rightsquigarrow$ LIMINF $= -\infty$ LIMSUP $= +\infty$

c) $0 \le x \le y$ $\rightsquigarrow$ LIMINF $= -\infty$ LIMSUP $= +\infty$

d) $x>0 \quad y \le x^2$



$$f(\delta, -\sqrt{\delta-\delta^2}) = \frac{\delta}{\delta-\delta+\delta^2} = \frac{1}{\delta} \to +\infty$$

$$f(\delta, -\sqrt{\delta+\delta^2}) = \frac{\delta}{\delta-\delta-\delta^2} = -\frac{1}{\delta} \to -\infty$$

LIMINF $= -\infty$ LIMSUP $= +\infty$

12) $\lim_{(x,y)\to(0,0)} \dfrac{\sqrt{x^2+|y|}}{|x|+|y|}$

$$\frac{\sqrt{x^2+|y|}}{|x|+|y|} = \frac{\sqrt{\cos^2\theta+|\sin\theta|/\rho}}{|\cos\theta|+|\sin\theta|} \to \begin{cases} +\infty & \sin\theta \neq 0 \\ 1 & \sin\theta = 0 \end{cases}$$

$|\cos\theta|+|\sin\theta| = m > 0$

a) $(x,y) \in \mathbb{R}^2$ LIMINF = 1 LIMSUP = $+\infty$

b) $x>0, y>0$ LIMINF = LIMSUP = $+\infty$

c) $0 \le x \le y$ LIMINF = LIMSUP = $+\infty$

d) $x>0$ $y \le x^2$ LIMINF = 1 LIMSUP = $+\infty$