BY GIMUSI 03/11/15

Serie 5

Argomenti: convergenza di serie a termini di segno qualunque **Difficoltà:** ⋆⋆⋆⋆

Prerequisiti: tutti i criteri di convergenza delle serie

Stabilire per quali valori del parametro reale $\alpha > 0$ le seguenti serie numeriche convergono.

	a) Serie	α	b) Serie	α
1)	$\displaystyle\sum_{n=1}^{\infty} \frac{\alpha^n}{n^\alpha}$	$0<\alpha<1$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)^\alpha}$	$\alpha>0$
2)	$\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+\alpha n^2}$	$\alpha>0$	$\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+\cos n)^\alpha}$	$\alpha>1$
3)	$\displaystyle\sum_{n=0}^{\infty} e^{n-n^\alpha}$	$\alpha>1$	$\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^\alpha}$	—
4)	$\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n}-\sin\frac{1}{n}\right)^\alpha$	$\alpha>1/3$	$\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n}-\sin\frac{1}{n}\right)^\alpha$	$\alpha>1$
5)	$\displaystyle\sum_{n=1}^{\infty} \left(\sinh\frac{3}{n}-\sin\frac{\alpha}{n}\right)$	$\alpha=3$	$\displaystyle\sum_{n=0}^{\infty} \frac{\alpha^n}{\cosh n}$	$\alpha<3$
6)	$\displaystyle\sum_{n=0}^{\infty} \alpha^n \arctan(2^n)$	$\alpha<1$	$\displaystyle\sum_{n=0}^{\infty} n^\alpha \arcsin\left(\frac{n^2}{2^n}\right)$	$\forall\alpha$
7)	$\displaystyle\sum_{n=1}^{\infty} \frac{\log(1+3^n)}{n^\alpha}$	$\alpha>2$	$\displaystyle\sum_{n=2}^{\infty} \frac{n+\cos n}{n^2\sqrt{\log^\alpha n+3}}$	$\alpha>2$
8)	$\displaystyle\sum_{n=0}^{\infty} \log^\alpha\left(\frac{n^2+2}{n^2+1}\right)$	$\alpha>1/2$	$\displaystyle\sum_{n=0}^{\infty} \log\left(\frac{n^2+2}{n^\alpha+1}\right)$	$\alpha=2$
9)	$\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{n+3}-1\right)^\alpha$	$\alpha>1$	$\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n^\alpha}\right)^{-n}$	$\alpha<1$
10)	$\displaystyle\sum_{n=0}^{\infty} \frac{\alpha^n}{1+\alpha^{3n}}$	$\alpha\neq1$	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-n^\alpha}$	$\alpha>1$
11)	$\displaystyle\sum_{n=1}^{\infty} \left(1-\sinh\frac{1}{n}\right)^{n^\alpha}$	$\alpha>1$	$\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[n]{2^n+n^2}-2}{\alpha^n}$	$\alpha>1/2$

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1.a) $\sum_{n=1}^{\infty} \frac{\alpha^n}{n^\alpha}$ CONVERGE $0<\alpha<1$

$$a_n = \frac{\alpha^n}{n^\alpha} \geq 0 \quad \forall n$$

$$\frac{a_{n+1}}{a_n} = \frac{\alpha^{n+1}}{\alpha^n} \cdot \frac{n^\alpha}{(n+1)^\alpha} = \alpha \left(\frac{n}{n+1}\right)^\alpha \to \alpha \begin{cases} >1 & \text{DIVERGE} \\ <1 & \text{CONVERGE} \end{cases}$$

$$\alpha = 1 \quad \sum_{n=1}^{\infty} \frac{\alpha^n}{n^\alpha} = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

1.b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)^\alpha}$ CONVERGE $\forall \alpha > 0$ (PER LEIBNITZ)

$$\alpha_n = \frac{1}{(n+2)^\alpha} \quad \begin{cases} \alpha_n \geq 0 \\ \alpha_{n+1} \leq \alpha_n & \forall \alpha > 0 \\ \alpha_n \to 0 \end{cases}$$

2.a) $\sum_{n=0}^{\infty} \frac{1}{1+\alpha n^2}$ CONVERGE $\forall \alpha > 0$

$$a_n \geq 0 \quad b_n = \frac{1}{n^2} \quad \frac{a_n}{b_n} = \frac{n^2}{1+\alpha n^2} \to \frac{1}{\alpha}$$

2.b) $\sum_{n=0}^{\infty} \frac{1}{(n+\cos n)^\alpha}$ CONVERGE $\forall \alpha > 1$

$$a_n \geq 0 \text{ DEF.} \quad b_n = \frac{1}{n^\alpha} \quad \frac{a_n}{b_n} = \left(\frac{n}{n+\cos n}\right)^\alpha \to 1$$

3.a) $\sum_{n=0}^{\infty} e^{n-n^\alpha}$ CONVERGE $\forall \alpha > 1$

$$\text{COND. NEC.} \quad e^{n-n^\alpha} \to 0 \Rightarrow n-n^\alpha < 0 \quad n^\alpha > n \quad \alpha > 1$$

$$a_n \geq 0 \quad b_n = \frac{1}{n^2} \quad \alpha > 1 \quad \frac{a_n}{b_n} = e^n \frac{n^2}{e^{n^\alpha}} \to 0$$

3.b) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\alpha}}$ NON CONVERGE

$a_n \geq 0 \quad b_n = \frac{1}{n} \qquad \frac{a_n}{b_n} = \frac{n}{(\log n)^{\alpha}} \to +\infty$

5.a) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\frac{1}{n}\right)^{\alpha}$ CONVERGE $\forall \alpha > 1/3$

$a_n \geq 0 \quad b_n = \frac{1}{n^{3\alpha}} \quad \frac{a_n}{b_n} = \left(n^2 - n^2 \sin\frac{1}{n}\right)^{\alpha} =$

$= \left(n^2 - n^3\left(\frac{1}{n} - \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)\right)\right)^{\alpha} = \left(\frac{1}{6} + o(1)\right)^{\alpha} \to \frac{1}{6^{\alpha}}$

5.b) $\sum_{n=1}^{\infty} \left(\frac{2}{n} - \sin\frac{1}{n}\right)^{\alpha}$ CONVERGE $\forall \alpha > 1$

$a_n \geq 0 \quad b_n = \frac{1}{n^{\alpha}} \quad \frac{a_n}{b_n} = \left(2 - n \sin\frac{1}{n}\right)^{\alpha} =$

$= \left(2 - n\left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right)\right)^{\alpha} = \left(1 + o(1)\right)^{\alpha} \to 1$

5.a) $\sum_{n=1}^{\infty} \left(\sinh\frac{3}{n} - \sin\frac{\alpha}{n}\right)$ CONVERGE PER $\alpha = 3$

$\sinh\frac{3}{n} = \frac{3}{n} + \frac{1}{2n^3} + o\left(\frac{1}{n^3}\right) \qquad \sin\frac{\alpha}{n} = \frac{\alpha}{n} - \frac{\alpha}{6n^3} + o\left(\frac{1}{n^3}\right)$

$\sinh\frac{3}{n} - \sin\frac{\alpha}{n} = \frac{3-\alpha}{n} + \left(\frac{1}{2} + \frac{\alpha}{6}\right)\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$

5.b) $\sum_{n=0}^{\infty} \frac{\alpha^n}{\cosh n}$ CONVERGE PER $\alpha < e$

$a_n \geq 0 \quad b_n = \frac{\alpha^n}{e^n} \qquad \frac{a_n}{b_n} = \frac{e^n}{\cosh n} \to 2$

$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{\alpha^n}{e^n} = \sum_{n=0}^{\infty} (\alpha/e)^n$ CONVERGE PER $|\alpha| < e$

6.a) $\sum_{n=0}^{\infty} \alpha^n \operatorname{ARCTAN}(2^n)$ CONVERGE PER $\alpha < 1$

$a_n \geq 0 \quad b_n = \alpha^n \quad \frac{a_n}{b_n} = \operatorname{ARCTAN}(2^n) \to \pi/2$

$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \alpha^n$ CONVERGE PER $|\alpha| < 1$

6.b) $\sum_{n=0}^{\infty} n^\alpha \operatorname{ARCSIN}\left(\frac{n^2}{2^n}\right)$ CONVERGE $\forall \alpha$

$a_n \geq 0 \quad b_n = \frac{n^{2+\alpha}}{2^n} \quad \frac{a_n}{b_n} = \frac{2^n}{n^2} \operatorname{ARCSIN}\left(\frac{n^2}{2^n}\right) =$

$= \frac{2^n}{n^2}\left(\frac{n^2}{2^n} + o\left(\frac{n^2}{2^n}\right)\right) \to 1$

$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{n^{2+\alpha}}{2^n} < +\infty \quad \forall \alpha \in \mathbb{R}$

$c_n = \frac{1}{n^2} \quad \frac{b_n}{c_n} = \frac{n^{4+\alpha}}{2^n} \to 0$

7.a) $\sum_{n=1}^{\infty} \frac{\log(1+3^n)}{n^\alpha}$ CONVERGE $\forall \alpha > 2$

$a_n \geq 0 \quad b_n = \frac{1}{n^{\alpha-1}} \quad \frac{a_n}{b_n} = \frac{\log(1+3^n)}{n} \to \log 3$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha-1}}$ CONVERGE PER $\alpha > 2$

7.b) $\sum_{n=2}^{\infty} \frac{n + \cos n}{n^2 \sqrt{\log^\alpha n + 3}}$ CONVERGE $\forall \alpha > 2$

$a_n \geq 0 \quad b_n = \frac{1}{n \log^{\alpha/2} n} \quad \frac{a_n}{b_n} = \frac{(n + \cos n)\log^{\alpha/2} n}{n\sqrt{\log^\alpha n + 3}} =$

$$= \frac{n \log^{\alpha/2} n}{n \log^{\alpha/2} n} \frac{\left(1+\frac{\cos n}{n}\right)}{\sqrt{1+3/\log^{\alpha} n}} \to 1$$

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n \log^{\alpha/2} n} \quad \text{CONV.} \quad \forall \alpha > 2$$

8.a) $\displaystyle\sum_{n=0}^{\infty} \log^{\alpha}\left(\frac{n^2+2}{n^2+1}\right)$ CONVERGE $\forall \alpha > 1/2$

$$a_n \geq 0 \quad b_n = \frac{1}{n^{2\alpha}} \quad \frac{a_n}{b_n} = \left(n^2 \log\left(1+\frac{1}{n^2+1}\right)\right)^{\alpha} =$$

$$= \left(n^2\left(\frac{1}{n^2+1} + o\left(\frac{1}{n^2}\right)\right)\right)^{\alpha} = \left(\frac{n^2}{1+n^2} + o(1)\right)^{\alpha} \to 1$$

8.b) $\displaystyle\sum_{n=0}^{\infty} \log\left(\frac{n^2+2}{n^{\alpha}+1}\right)$ CONVERGE PER $\alpha = 2$

COND. NEC. $\log\left(\frac{n^2+2}{n^{\alpha}+1}\right) \to 0 \Rightarrow \frac{n^2+2}{n^{\alpha}+1} \to 1 \Rightarrow \alpha = 2$

$$\alpha = 2 \quad \sum_{n=0}^{\infty} \log\left(\frac{n^2+2}{n^2+1}\right) < +\infty \quad (\text{VOL. } 8.a)$$

9.a) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{n+3} - 1\right)^{\alpha}$ CONVERGE $\forall \alpha > 1$

$$\sqrt[n]{n+3} = e^{\frac{1}{n}\log(n+3)} = e^{\frac{1}{n}(\log n + o(1))} = 1 + \frac{\log n}{n} + o\left(\frac{\log n}{n}\right)$$

$$a_n \geq 0 \quad b_n = \left(\frac{\log n}{n}\right)^{\alpha} \quad \frac{a_n}{b_n} = \left(1 + o(1)\right)^{\alpha} \to 1$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{\log n}{n}\right)^{\alpha} < +\infty \quad \forall \alpha > 1$$

$$c_n = \frac{1}{n^{(\alpha+1)/2}} \qquad \frac{b_n}{c_n} = \left(n^{\frac{\alpha+1}{2\alpha}} \frac{\log n}{n} \right)^{\alpha} \to \begin{cases} +\infty & \frac{\alpha+1}{2\alpha} \geq 1 \\ 0 & \frac{\alpha+1}{2\alpha} < 1 \end{cases}$$

$$\begin{cases} \frac{\alpha+1}{2\alpha} \geq 1 \quad 2\alpha \leq \alpha+1 \quad \alpha \leq 1 \quad \Rightarrow \sum c_n = +\infty \quad \Rightarrow \sum b_n = +\infty \\ \frac{\alpha+1}{2\alpha} < 1 \quad 2\alpha > \alpha+1 \quad \alpha > 1 \quad \Rightarrow \sum c_n < +\infty \quad \Rightarrow \sum b_n < +\infty \end{cases}$$

9.b) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n^\alpha}\right)^{-n}$ CONVERGE PER $\alpha < 1$

$$\left(1 + \frac{1}{n^\alpha}\right)^{-n} = e^{-n \log\left(1 + \frac{1}{n^\alpha}\right)} = e^{-n\left(\frac{1}{n^\alpha} + o\left(\frac{1}{n^\alpha}\right)\right)} \to \begin{cases} 1/e & \alpha = 1 \\ 1 & \alpha > 1 \end{cases}$$

$\underline{\alpha < 1}$

SIA $N \in \mathbb{N}$ T.C. $\alpha N \geq 1$ (ES. $N = \lfloor 1/\alpha \rfloor + 1$)

$$\log\left(1 + \frac{1}{n^\alpha}\right) = \frac{1}{n^\alpha} - \frac{1}{2n^{2\alpha}} + \dots + \frac{(-1)^{N+1}}{N n^{K\alpha}} + o\left(\frac{1}{n^{K\alpha}}\right)$$

$$\left(1 + \frac{1}{n^\alpha}\right)^{-n} = e^{-n \log\left(1 + \frac{1}{n^\alpha}\right)} =$$

$$= e^{-n^{1-\alpha} + n^{1-2\alpha}/2 + \dots + (-1)^N n^{1-N\alpha}/K + o(n^{1-N\alpha})} \to 0$$

CONDIZIONE NECESS. OK

$a_n \geq 0 \qquad b_n = \frac{1}{e^{n^{(1-\alpha)/2}}} \qquad \frac{a_n}{b_n} \to 0$

$\sum b_n < +\infty$ PER C.A. CON $c_n = \frac{1}{n^2} \geq 0 \qquad \frac{b_n}{c_n} \to 0$

$\Rightarrow \sum a_n$ CONVERGE $\forall \alpha < 1$

10.a) $\displaystyle\sum_{n=0}^{\infty} \frac{\alpha^n}{1+\alpha^{3n}}$ CONVERGE PER $\alpha \neq 1$

COND. NEC. $\displaystyle \frac{\alpha^n}{1+\alpha^{3n}} = \frac{1}{\alpha^{-n}+\alpha^{2n}} \to \begin{cases} 0 & \alpha \neq 1 \\ 1/2 & \alpha = 1 \end{cases}$

$\underline{\alpha > 1}$

$a_n \geq 0 \quad b_n = \left(\frac{1}{\alpha}\right)^n \quad \sum b_n < +\infty \quad \displaystyle\frac{a_n}{b_n} = \frac{\alpha^{2n}}{1+\alpha^{3n}} \to 0$

$\underline{\alpha < 1}$

$a_n \geq 0 \quad b_n = \alpha^n \quad \sum b_n < +\infty \quad \displaystyle\frac{a_n}{b_n} = \frac{1}{1+\alpha^{3n}} \to 1$

10.b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-n^\alpha}$ CONVERGE PER $\alpha > 1$

COND. NEC. $\displaystyle \frac{1}{2\sqrt{n}-n^\alpha} \to \begin{cases} 0^+ & \alpha \leq 1/2 \\ 0^- & \alpha > 1/2 \end{cases}$

$\underline{\alpha \leq 1/2}$

$a_n \geq 0 \quad b_n = 1/\sqrt{n} \quad \sum b_n = +\infty \quad \displaystyle\frac{a_n}{b_n} \to 1/2$

$\underline{\alpha > 1/2}$

$-a_n \geq 0 \quad b_n = 1/n^\alpha \quad \sum b_n < +\infty \ (\alpha > 1) \quad \displaystyle\frac{-a_n}{b_n} \to 1$

11.a) $\displaystyle\sum_{n=1}^{\infty} \left(1-\sinh(1/n)\right)^{n^\alpha}$ CONVERGE PER $\alpha > 1$

$$\left(1-\sinh(1/n)\right)^{n^\alpha} = e^{n^\alpha \log(1-\sinh(1/n))} =$$

$$= e^{n^\alpha(-1/n + o(1/n))} = e^{-n^{\alpha-1}+o(n^{\alpha-1})} \to \begin{cases} 1/e & \alpha = 1 \\ 1 & \alpha < 1 \end{cases}$$

$\underline{\alpha > 1}$

SIA $k \in \mathbb{N}$ T.C. $k \geq \alpha$

$$\lg\left(1-\sinh(1/n)\right) = \sum_{i=1}^{K} a_i/n^i + o(1/n^K) \qquad a_1 = -1$$

$$\left(1-\sinh(1/n)\right)^{n^\alpha} = e^{n^\alpha \lg(1-\sinh(1/n))} = e^{\sum_{i=1}^{K} a_i n^{\alpha-i} + o(1/n^{K-\alpha})} \to$$

$\to 0$ CONDIZIONE NECESS. OK

$$a_n \geq 0 \quad b_n = \frac{1}{e^{n^{(\alpha-1)/2}}} \qquad \frac{a_n}{b_n} \to 0$$

$$\sum b_n < +\infty \quad \text{PER C.A. CON } c_n = \frac{1}{n^2} \geq 0 \quad \frac{b_n}{c_n} \to 0$$

$\Rightarrow \sum a_n$ CONVERGE $\forall \alpha > 1$

11.b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[n]{2^n+n^2}-2}{\alpha^n}$ CONVERGE PER $\alpha > 1/2$

$$\sqrt[n]{2^n+n^2} - 2 = 2\left(\left(1+n^2/2^n\right)^{1/n} - 1\right) = 2n/2^n + o(n/2^n)$$

$$\left(1+n^2/2^n\right)^{1/n} = e^{\frac{1}{n}\lg(1+n^2/2^n)} = e^{n/2^n + o(n/2^n)} =$$

$$= 1 + n/2^n + o(n/2^n)$$

TERMINE CHE $\to +\infty$ PER $\alpha \leq 1/2$

$$\frac{\sqrt[n]{2^n+n^2}-2}{\alpha^n} = \frac{2n}{(2\alpha)^n} + o\left(\frac{n}{(2\alpha)^n}\right) \to \begin{cases} 0 & 2\alpha > 1 \quad \alpha > 1/2 \\ +\infty & 2\alpha \leq 1 \end{cases}$$

ANDREBBE DIMOSTRATO MEGLIO

<u>$\alpha > 1/2$</u>

$$a_n \geq 0 \quad b_n = \frac{1}{n^2} \qquad \frac{a_n}{b_n} \to 0$$