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Serie 3

Argomenti: convergenza di serie a termini di segno costante **Difficoltà:** ⋆⋆⋆

Prerequisiti: criterio del confronto asintotico (inclusi casi limite), sviluppi di Taylor

Stabilire se le seguenti serie numeriche sono convergenti oppure no.

	Serie a)	Conv.?	Serie b)	Conv.?	Serie c)	Conv.?
1)	$\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$	SÍ	$\sum_{n=2}^{\infty} \frac{1}{n^{\log n}}$	SÍ	$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$	SÍ
2)	$\sum_{n=0}^{\infty} \frac{1}{2^{\sqrt{n}}}$	SÍ	$\sum_{n=1}^{\infty} \frac{1}{2^{\log n}}$	NO	$\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{n}}}$	SÍ
3)	$\sum_{n=1}^{\infty} \frac{\log n}{n}$	NO	$\sum_{n=2}^{\infty} \frac{1}{n \log n}$	NO	$\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$	SÍ
4)	$\sum_{n=2}^{\infty} \frac{1}{n \log^2 n}$	SÍ	$\sum_{n=1}^{\infty} \frac{\log^7 n}{n^2}$	SÍ	$\sum_{n=0}^{\infty} \frac{n^3}{2^{\sqrt{n}}}$	SÍ

Stabilire se le seguenti serie hanno i termini definitivamente positivi o negativi, quindi studiarne la convergenza.

	Serie a)	P/N	Conv.?	Serie b)	P/N	Conv.?
5)	$\sum_{n=1}^{\infty} \frac{n + \cos n}{n^2 \log n + \sin n}$	P	SÍ	$\sum_{n=1}^{\infty} \frac{n^2 + 3\sqrt[3]{n}}{n^2 \log^2 n + 4}$	P	NO
6)	$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sinh \frac{1}{n}\right)$	N	SÍ	$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \sin \frac{1}{\sqrt{n}}\right)$	P	SÍ
7)	$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin \frac{1}{n^2}\right)$	P	NO	$\sum_{n=1}^{\infty} \left(\cosh \frac{1}{\sqrt{n}} - \cos \frac{1}{\sqrt{n}}\right)$	P	NO
8)	$\sum_{n=1}^{\infty} (\log(n^4 - 4) - 4 \log n)$	N	SÍ	$\sum_{n=1}^{\infty} \left(\cosh \frac{1}{n^2} - \cos \frac{1}{n}\right)$	P	SÍ
9)	$\sum_{n=1}^{\infty} \left(\frac{\pi}{2} - \arctan n\right)$	P	NO	$\sum_{n=1}^{\infty} n^{(1-3n^2)/(n^2+7)}$	P	SÍ
10)	$\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$	P	NO	$\sum_{n=1}^{\infty} \left(\sqrt[n]{n^3 + 3^n} - 3\right)$	P	SÍ

1/7

1.a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\lg n)^n}$ CONVERGE

$$\sqrt[n]{a_n} = \frac{1}{\lg n} \to 0$$

1.b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{\lg n}}$ CONVERGE

$$b_n = \frac{1}{n^2} \qquad \frac{a_n}{b_n} = n^{2-\lg n} = e^{\overbrace{(2-\lg n)}^{\to -\infty}\lg n} \to 0$$

1.c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\lg n)^{\lg n}}$ CONVERGE

CRITERIO DI CONDENSAZIONE DI CAUCHY

$$\begin{cases} a_n \geq 0 \\ a_{n+1} \leq a_n \\ a_n \to 0 \end{cases} \quad \sum a_n < +\infty \iff \sum 2^n a_{2^n} < +\infty$$

$$\frac{1}{(\lg n)^{\lg n}} \rightsquigarrow 2^n \frac{1}{(\lg 2^n)^{\lg 2^n}} = \frac{2^n}{(n \lg 2)^{n \lg 2}} = b_n$$

$$\sum_{n=2}^{\infty} \frac{1}{(\lg n)^{\lg n}} < +\infty \iff \sum_{n=2}^{\infty} \frac{2^n}{(n\lg 2)^{n \lg 2}} < +\infty$$

$$\sqrt[n]{b_n} = \frac{2}{(n \lg 2)^{\lg 2}} \to 0 \Rightarrow \sum_{n=2}^{\infty} b_n < +\infty \Rightarrow \sum_{n=2}^{\infty} a_n < +\infty$$

2.a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{\sqrt{n}}}$ CONVERGE

$$b_n = \frac{1}{n^2} \qquad \frac{a_n}{b_n} = \frac{n^2}{2^{\sqrt{n}}} = \frac{(\sqrt{n})^4}{2^{\sqrt{n}}} \to 0$$

2.b) $\sum_{n=1}^{\infty} \frac{1}{2^{\log n}}$ NON CONVERGE

$$b_n = \frac{1}{n} \qquad \frac{a_n}{b_n} = \frac{n}{2^{\log n}} = \frac{e^{\log n}}{2^{\log n}} = \left(\frac{e}{2}\right)^{\log n} \to +\infty$$

2.c) $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{n}}}$ CONVERGE

$$b_n = \frac{1}{n^2} \qquad \frac{a_n}{b_n} = \frac{n^2}{n^{\sqrt{n}}} = \frac{1}{n^{(\sqrt{n}-2)}} \to 0$$

3.a) $\sum_{n=1}^{\infty} \frac{\log n}{n}$ NON CONVERGE

$$b_n = \frac{1}{n} \qquad \frac{a_n}{b_n} = \log n \to +\infty$$

3.b) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ NON CONVERGE

CRITERIO DI CONDENSAZIONE DI CAUCHY

$$\begin{cases} a_n \geq 0 \\ a_{n+1} \leq a_n \\ a_n \to 0 \end{cases} \quad \sum a_n < +\infty \iff \sum 2^n a_{2^n} < +\infty$$

$$\frac{1}{n \log n} \rightsquigarrow \frac{\cancel{2^n}}{\cancel{2^n} \log 2^n} = \frac{1}{n}$$

3.c) $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$ CONVERGE

$$b_n = \frac{1}{n^2} \qquad \frac{a_n}{b_n} = \frac{1}{\log n} \to 0$$

5.a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log^2 n}$ CONVERGE

CRITERIO DI CONDENSAZIONE DI CAUCHY

$$\begin{cases} a_n \geq 0 \\ a_{n+1} \leq a_n \\ a_n \to 0 \end{cases} \quad \sum a_n < +\infty \iff \sum 2^n a_{2^n} < +\infty$$

$$\frac{1}{n \log^2 n} \rightsquigarrow \frac{\cancel{2^n}}{\cancel{2^n} \log^2 2^n} = \frac{1}{n^2}$$

5.b) $\displaystyle\sum_{n=1}^{\infty} \frac{\log^7 n}{n^2}$ CONVERGE

$$b_n = \frac{1}{n^{3/2}} \qquad \frac{a_n}{b_n} = \frac{\log^7 n}{\sqrt{n}} \to 0$$

5.c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{2^{\sqrt{n}}}$ CONVERGE

$$b_n = \frac{1}{n^3} \qquad \frac{a_n}{b_n} = \frac{n^6}{2^{\sqrt{n}}} = \frac{(\sqrt{n})^{12}}{2^{\sqrt{n}}} \to 0$$

5.a) $\displaystyle\sum_{n=1}^{\infty} \frac{n + \cos n}{n^2 \log n + \sin n}$ P CONVERGE

DEFINIT. $\begin{cases} n + \cos n > 0 & \forall n \geq 1 \\ n^2 \log n + \sin n > 0 & \forall n \geq 2 \end{cases} \rightsquigarrow a_n > 0$ DEF.

$$b_n = \frac{1}{n \log n} \qquad \frac{a_n}{b_n} = \frac{n^2 \log n}{n^2 \log n} \, \frac{1 + \frac{\cos n}{n}}{1 + \frac{\sin n}{n^2 \log n}} \to 1$$

5.b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 3\sqrt[3]{n}}{n^2 \log^2 n + 5}$ P NON CONVERGE

$$b_n = \frac{1}{\log^2 n} \qquad \sum_{n=2}^{\infty} b_n = +\infty \qquad \frac{a_n}{b_n} = \frac{n^2 \log^2 n + 3\sqrt[3]{n} \log^2 n}{n^2 \log^2 n + 5} =$$

$$= \frac{n^2 \log^2 n}{n^2 \log^2 n} \, \frac{1 + \frac{3\sqrt[3]{n} \cancel{\log^2 n}}{n^2 \cancel{\log^2 n}}}{1 + \frac{5}{n^2 \log^2 n}} \to 1$$

6.a) $\sum_{n=1}^{\infty} \frac{1}{n} - \sinh \frac{1}{n}$ N CONVERGE

$$\sinh \frac{1}{n} = \frac{1}{n} + \frac{1}{6}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right) \qquad \frac{1}{n} - \sinh\frac{1}{n} = -\frac{1}{6}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right) \le 0$$

$$b_n = -\frac{1}{n^3} \qquad -\infty < \sum_{n=1}^{\infty} b_n \le 0 \qquad \frac{a_n}{b_n} = \frac{-\frac{1}{6}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)}{-1/n^3} \to 1/6$$

6.b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \sin \frac{1}{\sqrt{n}}$ P CONVERGE

$$\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}} + \frac{1}{6}\frac{1}{\sqrt{n^3}} + o\left(\frac{1}{\sqrt{n^3}}\right) = \frac{1}{6}\frac{1}{\sqrt{n^3}} + o\left(\frac{1}{\sqrt{n^3}}\right) \ge 0$$

$$b_n = \frac{1}{\sqrt{n^3}} \qquad \sum_{n=1}^{\infty} b_n < +\infty \qquad \frac{a_n}{b_n} = \frac{\frac{1}{6}\frac{1}{\sqrt{n^3}} + o\left(\frac{1}{\sqrt{n^3}}\right)}{1/\sqrt{n^3}} \to \frac{1}{6}$$

7.a) $\sum_{n=1}^{\infty} \frac{1}{n} - \sin \frac{1}{n^2}$ P NON CONVERGE

$$\frac{\frac{1}{n} - \sin\frac{1}{n^2}}{1/n} \to 1 \implies \frac{1}{n} - \sin\frac{1}{n^2} \ge 0$$

$$b_n = \frac{1}{n} \qquad \sum_{n=1}^{\infty} b_n = +\infty \qquad \frac{a_n}{b_n} = \frac{\frac{1}{n} - \sin\frac{1}{n^2}}{1/n} \to 1$$

7.b) $\sum_{n=1}^{\infty} \cosh \frac{1}{\sqrt{n}} - \cos \frac{1}{\sqrt{n}}$ P NON CONVERGE

$$\cosh x \ge 1 \ge \cos x \quad \forall x \implies \cosh\frac{1}{\sqrt{n}} - \cos\frac{1}{\sqrt{n}} \ge 0$$

$$b_n = \frac{1}{n} \qquad \sum_{n=1}^{\infty} b_n = +\infty \qquad \frac{a_n}{b_n} = \frac{\cosh\frac{1}{\sqrt{n}} - \cos\frac{1}{\sqrt{n}}}{1/n} =$$

$$= \frac{\cancel{1} + \frac{1}{2}\frac{1}{n} - \cancel{1} + \frac{1}{2}\frac{1}{n} + o\left(\frac{1}{n}\right)}{1/n} \to 1$$

8.a) $\sum_{n=1}^{\infty} (\log(n^5-5) - 5\log n)$ N CONVERGE

$$\log(n^5-5) - 5\log n = \log\Big(\underbrace{1 - \frac{5}{n^5}}_{\to 1^-}\Big) \to 0^- \quad \Rightarrow a_n \leq 0 \text{ DEF.}$$

$$b_n = -\frac{1}{n^5} \qquad -\infty < \sum_{n=1}^{\infty} b_n \leq 0 \qquad \frac{a_n}{b_n} = \frac{-\frac{5}{n^5} + o\left(\frac{1}{n^5}\right)}{-1/n^5} \to 5$$

8.b) $\sum_{n=1}^{\infty} \cosh\frac{1}{n^2} - \cos\frac{1}{n}$ P CONVERGE

$$\cosh\frac{1}{n^2} - \cos\frac{1}{n} = \cancel{1} + \frac{1}{2n^4} - \cancel{1} + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \geq 0 \text{ DEF.}$$

$$b_n = \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} b_n < +\infty \qquad \frac{a_n}{b_n} = \frac{\frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)}{1/n^2} \to \frac{1}{2}$$

9.a) $\sum_{n=1}^{\infty} \frac{\pi}{2} - \text{ARCTAN } n$ P NON CONVERGE

$$\text{ARCTAN } n \leq \frac{\pi}{2} \Rightarrow a_n \geq 0 \quad \forall n$$

$$\text{ARCTAN } n = \frac{\pi}{2} - \text{ARCTAN}(1/n) \Rightarrow \frac{\pi}{2} - \text{ARCTAN } n = \text{ARCTAN}(1/n)$$

$$\sum_{n=1}^{\infty} \frac{\pi}{2} - \text{ARCTAN } n = \sum_{n=1}^{\infty} \text{ARCTAN}(1/n)$$

$$b_n = \frac{1}{n} \qquad \sum_{n=1}^{\infty} b_n = +\infty \qquad \frac{a_n}{b_n} = \frac{\frac{1}{n} + o\left(\frac{1}{n}\right)}{1/n} \to 1$$

9.b) $\sum_{n=1}^{\infty} n^{(1-3n^2)/(n^2+7)}$ P CONVERGE

$$a_n = n^{f(n)} \geq 0 \quad \forall n \geq 1 \qquad b_n = \frac{1}{n^3} \qquad \sum_{n=1}^{\infty} b_n < +\infty$$

$$\frac{a_n}{b_n} = n^{\frac{1-3n^2-3n^2-21}{n^2+7}} = n^{\frac{-20}{n^2+7}} = e^{-20\overbrace{\frac{\log n}{n^2+7}}^{\to 0}} \to 1$$

10.a) $\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$ P NON CONVERGE

$\sqrt{n+1} \geq \sqrt{n} \Rightarrow a_n \geq 0 \quad b_n = \frac{1}{\sqrt{n}} \quad \sum_{n=1}^{\infty} b_n = +\infty$

$$\frac{a_n}{b_n} = \frac{\sqrt{n+1} - \sqrt{n}}{1/\sqrt{n}} = n\left(\left(1+\frac{1}{n}\right)^{1/2} - 1\right) =$$

$$= n\left(\cancel{1} + \frac{1}{2n} + o\left(\frac{1}{n}\right) - \cancel{1}\right) \to \frac{1}{2}$$

10.b) $\sum_{n=1}^{\infty} \sqrt[n]{n^3+3^n} - 3$ P CONVERGE

$\sqrt[n]{n^3+3^n} \geq \sqrt[n]{3^n} = 3 \quad a_n \geq 0$

$$\sqrt[n]{n^3+3^n} = e^{\frac{\log(n^3+3^n)}{n}} = e^{\frac{\log(1+n^3/3^n) + n\log 3}{n}} =$$

$$= e^{\log 3 + \frac{1}{n}\left(\frac{n^3}{3^n} + o\left(\frac{n^3}{3^n}\right)\right)} = 3\left(1 + \frac{n^2}{3^n} + o\left(\frac{n^2}{3^n}\right)\right)$$

$b_n = \frac{n^2}{3^n} \quad \sum_{n=1}^{\infty} b_n < +\infty \quad \left(\frac{n^2}{3^n} \leq \frac{n^2}{n^4} = \frac{1}{n^2}\right)$

$$\frac{a_n}{b_n} = \frac{3\frac{n^2}{3^n} + o\left(\frac{n^2}{3^n}\right)}{n^2/3^n} \to 3$$