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Linguaggio degli infinitesimi 1 – o piccolo

Argomenti: o piccolo **Difficoltà:** ⋆⋆⋆

Prerequisiti: definizione di o piccolo, limiti notevoli, sviluppini

Stabilire se le seguenti proposizioni sono vere o false (gli o piccolo si intendono per $x \to 0^+$).

	a) Proposizione	V/F	b) Proposizione	V/F	c) Proposizione	V/F
1)	$x^2 = o(x)$	V	$x^2 = o(\sqrt{x})$	V	$x^2 = o(x\sqrt{x})$	V
2)	$\sin(3x^2) = o(x)$	V	$\sin(3x^2) = o(7x)$	V	$\sin(3x^2) = o(2x^2)$	F
3)	$\cos x = o(1)$	F	$\sin x = o(\cos x)$	V	$\cos x = o(\sin x)$	F
4)	$\tan^2 x = o(\arctan x)$	V	$\tan x = o(\arctan^2 x)$	F	$\tan(x^2) = o(\sin x)$	V
5)	$x = o(\log x)$	V	$x = o(x \log x)$	V	$x \log x = o(x)$	F
6)	$\sin x = o(\sqrt{x})$	V	$\sin x = o(x\sqrt{x})$	F	$x = o(\sin\sqrt{x})$	V
7)	$\sin(2x) = x + o(x)$	F	$\sin x = 2x + o(x)$	F	$\sin(2x) = x + o(2x)$	F
8)	$\sin(x^2) = x^2 + o(x^2)$	V	$\sin(x^2) = x^2 + o(x)$	V	$\sin(x^2) = x^2 + o(1)$	V
9)	$\sin^2 x = x^2 + o(x^2)$	V	$\sin^2 x = x^2 + o(x)$	V	$\sin(x^2) = o(x \sin x)$	F

	a) Proposizione	V/F	b) Proposizione	V/F
10)	$\sin x + \cos x = x + o(x)$	F	$\sin(2e^x - \cos x - 1) = o(x)$	F
11)	$e^x - \cos x = x + o(x)$	V	$e^{2x} - \cos(3x) = -x + o(x)$	F
12)	$\log(1 + \tan(3x \sin x)) = 3x^2 + o(x^2)$	V	$\log(\cos(4x)) = -8\sin^2 x + o(x^2)$	V

Stabilire se le seguenti implicazioni sono vere o false per ogni funzione $f(x)$ (tutti gli o piccolo si intendono per $x \to 0$ e tutte le funzioni $f(x)$ si intendono per semplicità definite su tutto $\mathbb{R}$).

13) • Se $f(x) = x + o(x^4)$, allora $f^2(x) = x^2 + o(x^8)$. F

14) • Se $f(x) = x + o(x^4)$, allora $f^2(x) = x^2 + o(x^5)$. V

15) • Se $f^2(x) = x^4 + o(x^8)$, allora $|f(x)| = x^2 + o(x^8)$. F

16) • Se $f^2(x) = x^4 + o(x^8)$, allora $|f(x)| = x^2 + o(x^6)$. ~~F~~ V

17) • Se $f(x) = x + o(x)$, allora $f(f(x)) = x + o(x)$. V

18) • Se $f(x) = x^3 + o(x^5)$, allora $f(f(x)) = x^9 + o(x^{11})$. V

19) • Se $f(x) = x^2 + o(f(x))$, allora $\cancel{f(f}(x)) = x^2 + o(x^2)$. ~~F~~ V

$f(x)$

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DEFINIZIONE DI "O PICCOLO"

SIANO $\begin{cases} f: D \to \mathbb{R} \quad g: D \to \mathbb{R} \\ x_0 \in \overline{\mathbb{R}} \quad \text{P.TO NEL QUALE HA SENSO FARE I LIMITI} \end{cases}$

DEF. f É "O PICCOLO" DI g PER $x \to x_0$

$$f(x) = o(g(x)) \quad x \to x_0$$

SE $\exists\, \omega : D \to \mathbb{R}$ S.C.

$$f(x) = g(x) \cdot \omega(x) \quad \forall x \in D$$

$$\lim_{x \to x_0} \omega(x) = 0$$

1.a) $x^2 = o(x)$ V

$x^2 = x \cdot x \qquad \omega(x) = x \to 0$

1.b) $x^2 = o(\sqrt{x})$ V

$x^2 = \sqrt{x} \cdot \frac{x^2}{\sqrt{x}} = \sqrt{x} \cdot x\sqrt{x} \qquad \omega(x) = x\sqrt{x} \to 0$

1.c) $x^2 = o(x\sqrt{x})$ V

$x^2 = x\sqrt{x} \cdot \sqrt{x} \qquad \omega(x) = \sqrt{x} \to 0$

2.a) $\sin(3x^2) = o(x)$ V

$\sin(3x^2) = x \cdot \frac{\sin(3x^2)}{x} \qquad \omega(x) = \frac{\sin(3x^2)}{x}$

$\lim_{x\to 0^+} \omega(x) = \lim_{x\to 0^+} \frac{\sin(3x^2)}{x} = \lim_{x\to 0^+} \frac{\sin(3x^2)}{3x^2} \cdot 3x = 0$

2.b) $\sin(3x^2) = o(7x)$ V

$\sin(3x^2) = 7x \cdot \frac{\sin(3x^2)}{7x} \qquad \omega(x) = \frac{\sin(3x^2)}{7x} \to 0$

2.c) $\sin(3x^2) = o(2x^2)$ F

$\sin(3x^2) = 2x^2 \cdot \frac{\sin(3x^2)}{2x^2} \qquad \omega(x) = \frac{\sin(3x^2)}{2x^2}$

$\omega(x) = \frac{\sin(3x^2)}{2x^2} = \frac{\sin(3x^2)}{3x^2} \cdot \frac{3}{2} \to \frac{3}{2}$

3.a) $\cos(x) = o(1)$ F

$\cos(x) = 1 \cdot \cos(x) \qquad \omega(x) = \cos(x) \to 1$

3.b) $\sin x = o(\cos x)$ V

$$\sin x = \cos x \cdot \frac{\sin x}{\cos x} \qquad \omega(x) = \frac{\sin x}{\cos x} \to 0$$

3.c) $\cos x = o(\sin x)$ F

$$\cos x = \sin x \cdot \frac{\cos x}{\sin x} \qquad \omega(x) = \frac{\cos x}{\sin x} \to +\infty$$

5.a) $\tan^2 x = o(\arctan x)$ V

$$\tan^2 x = \arctan x \cdot \frac{\tan^2 x}{\arctan x} \qquad \omega(x) = \frac{\tan^2 x}{\arctan x}$$

$$\omega(x) = \frac{\tan^2 x}{\arctan x} = \frac{\tan^2 x}{x^2} \; \frac{x}{\arctan x} \cdot x \to 0$$

5.b) $\tan x = o(\arctan^2 x)$ F

$$\tan x = \arctan^2 x \cdot \frac{\tan x}{\arctan^2 x} \qquad \omega(x) = \frac{\tan x}{\arctan^2 x}$$

$$\omega(x) = \frac{\tan x}{\arctan^2 x} = \frac{\tan x}{x} \; \frac{x^2}{\arctan^2 x} \cdot \frac{1}{x} \to +\infty$$

5.c) $\tan(x^2) = o(\sin x)$ V

$$\tan(x^2) = \sin x \cdot \frac{\tan(x^2)}{\sin x} \qquad \omega(x) = \frac{\tan(x^2)}{\sin x}$$

$$\omega(x) = \frac{\tan(x^2)}{\sin x} = \frac{\tan(x^2)}{x^2} \cdot \frac{x}{\sin x} \; x \to 0$$

5.e) $x = o(\log x)$ V

$$x = \log x \cdot \frac{x}{\log x} \qquad \omega(x) = \frac{x}{\log x} \to 0$$

5.b) $x = o(x \log x)$ V

$x = x \log x \cdot \frac{1}{\log x}$ $\qquad \omega(x) = \frac{1}{\log x} \to 0$

5.c) $x \log x = o(x)$ F

$x \log x = x \cdot \log x$ $\qquad \omega(x) = \log x \to -\infty$

6.a) $\sin x = o(\sqrt{x})$ V

$\sin x = \sqrt{x} \cdot \frac{\sin x}{\sqrt{x}}$ $\qquad \omega(x) = \frac{\sin x}{\sqrt{x}} = \frac{\sin x}{x} \sqrt{x} \to 0$

6.b) $\sin x = o(x\sqrt{x})$ F

$\sin x = x\sqrt{x} \cdot \frac{\sin x}{x\sqrt{x}}$ $\qquad \omega(x) = \frac{\sin x}{x\sqrt{x}} = \frac{\sin x}{x} \frac{1}{\sqrt{x}} \to +\infty$

6.c) $x = o(\sin\sqrt{x})$ V

$x = \sin\sqrt{x} \cdot \frac{x}{\sin\sqrt{x}}$ $\qquad \omega(x) = \frac{x}{\sin\sqrt{x}} = \frac{\sqrt{x}}{\sin\sqrt{x}} \sqrt{x} \to 0$

7.a) $\sin(2x) = x + o(x)$ F

$\sin(2x) - x = x \cdot \frac{\sin(2x) - x}{x}$ $\qquad \omega(x) = \frac{\sin(2x) - x}{x}$

$\omega(x) = \frac{\sin(2x) - x}{x} = 2\frac{\sin(2x)}{2x} - 1 \to 1$

7.b) $\sin x = 2x + o(x)$ F

$\sin x - 2x = x \cdot \frac{\sin x - 2x}{x}$ $\qquad \omega(x) = \frac{\sin x - 2x}{x} \to -1$

7.c) $\sin(2x) = x + o(2x)$ F

$\sin(2x) - x = 2x \cdot \frac{\sin(2x) - x}{2x}$ $\qquad \omega(x) = \frac{\sin(2x) - x}{2x} \to 1/2$

8.a) $\sin(x^2) = x^2 + o(x^2)$ V

$$\sin(x^2) - x^2 = x^2 \cdot \frac{\sin(x^2) - x^2}{x^2} \qquad \omega(x) = \frac{\sin(x^2) - x^2}{x^2} \to 0$$

8.b) $\sin(x^2) = x^2 + o(x)$ V

$$\sin(x^2) - x^2 = x \cdot \frac{\sin(x^2) - x^2}{x} \qquad \omega(x) = \frac{\sin(x^2) - x^2}{x}$$

$$\omega(x) = \frac{\sin(x^2) - x^2}{x} = \frac{\sin(x^2) - x^2}{x^2} \cdot x \to 0$$

8.c) $\sin(x^2) = x^2 + o(1)$ V

$$\sin(x^2) - x^2 = 1 \cdot [\sin(x^2) - x^2] \qquad \omega(x) = \sin(x^2) - x^2 \to 0$$

9.a) $\sin^2 x = x^2 + o(x^2)$ V

$$\sin^2 x - x^2 = x^2 \cdot \frac{\sin^2 x - x^2}{x^2} \qquad \omega(x) = \frac{\sin^2 x - x^2}{x^2} \to 0$$

9.b) $\sin^2 x = x^2 + o(x)$ V

$$\sin^2 x - x^2 = x \frac{\sin^2 x - x^2}{x} \qquad \omega(x) = \frac{\sin^2 x - x^2}{x} \to 0$$

9.c) $\sin(x^2) = o(x \sin x)$ F

$$\sin(x^2) = x \sin x \cdot \frac{\sin(x^2)}{x \sin x} \qquad \omega(x) = \frac{\sin(x^2)}{x \sin x}$$

$$\omega(x) = \frac{\sin(x^2)}{x \sin x} = \frac{\sin(x^2)}{x^2} \frac{x}{\sin x} \to 1$$

10.a) $\sin x + \cos x = x + o(x)$ F

$$\sin x + \cos x - x = x \cdot \frac{\sin x + \cos x - x}{x} \qquad \omega(x) = \frac{\sin x + \cos x - x}{x}$$

$$\omega(x) = \frac{\sin x + \cos x - x}{x} = \frac{\sin x}{x} + \frac{\cos x}{x} - 1 \to +\infty$$

10.b) $\sin(2e^x - \cos x - 1) = o(x)$ F

$$\sin(2e^x-\cos x-1) = x \cdot \frac{\sin(2e^x-\cos x-1)}{x}$$

$$\omega(x) = \frac{\sin(2e^x-\cos x-1)}{x} = \frac{\sin(2e^x-\cos x-1)}{2e^x-\cos x-1}\,\frac{2e^x-\cos x-1}{x} =$$

$$= \overset{\to 1}{\frac{\sin(2e^x-\cos x-1)}{2e^x-\cos x-1}}\left(\overset{\to 2}{2\,\frac{e^x-1}{x}} + \overset{\to 0}{\frac{1-\cos x}{x^2}\,x}\right) \to 2$$

11.a) $e^x - \cos x = x + o(x)$ V

$$e^x-\cos x - x = x \cdot \frac{e^x-\cos x - x}{x} \qquad \omega(x) = \frac{e^x-\cos x - x}{x}$$

$$\omega(x) = \frac{e^x-\cos x - x}{x} = \overset{\to 1}{\frac{e^x-1}{x}} + \overset{\to 0}{\frac{1-\cos x}{x^2}\,x} - 1 \to 0$$

11.b) $e^{2x} - \cos(3x) = -x + o(x)$ F

$$e^{2x}-\cos(3x)+x = x \cdot \frac{e^{2x}-\cos(3x)+x}{x} \qquad \omega(x) = \frac{e^{2x}-\cos(3x)+x}{x}$$

$$\omega(x) = \frac{e^{2x}-\cos(3x)+x}{x} = \overset{\to 2}{2\,\frac{e^{2x}-1}{2x}} + \overset{\to 0}{\frac{1-\cos 3x}{3x^2}\cdot 3x} + 1 \to 3$$

12.a) $\log(1+\tan(3x\sin x)) = 3x^2 + o(x^2)$ V

$$\log(1+\tan(3x\sin x)) - 3x^2 = x^2 \cdot \frac{\log(1+\tan(3x\sin x)) - 3x^2}{x^2}$$

$$\omega(x) = \frac{\log(1+\tan(3x\sin x)) - 3x^2}{x^2} =$$

$$= \overset{\to 1}{\frac{\log(1+\tan(3x\sin x))}{\tan(3x\sin x)}}\;\overset{\to 1}{\frac{\tan(3x\sin x)}{3x\sin x}}\;\overset{\to 3}{\frac{3x\sin x}{x^2}} - 3 \to 0$$

12.b) $\log(\cos(4x)) = -8\sin^2 x + o(x^2)$ V

$$\log(\cos(4x)) + 8\sin^2 x = x^2 \cdot \frac{\log(\cos(4x)) + 8\sin^2 x}{x^2}$$

$$\omega(x) = \frac{\log(\cos(4x)) + 8\sin^2 x}{x^2} =$$

$$= \underbrace{\frac{\log(1+(\cos(4x)-1))}{\cos(4x)-1}}_{\to 1} \underbrace{\frac{\cos(4x)-1}{(4x)^2} 16}_{-8} + \underbrace{8\frac{\sin^2 x}{x^2}}_{+8} \to 0$$

Stabilire se le seguenti implicazioni sono vere o false per ogni funzione $f(x)$ (tutti gli o piccolo si intendono per $x \to 0$ e tutte le funzioni $f(x)$ si intendono per semplicità definite su tutto $\mathbb{R}$).

13) • Se $f(x) = x + o(x^4)$, allora $f^2(x) = x^2 + o(x^8)$. F

$$f^2(x) = (x + o(x^4))^2 = x^2 + 2x \cdot o(x^4) + o(x^4)^2$$

$$x \cdot o(x^4) = x \cdot x^4 \cdot \omega(x) = o(x^5)$$

$$o(x^4)^2 = x^8 \cdot \omega(x)^2 = o(x^8) = o(x^5)$$

$$\rightsquigarrow f^2(x) = x^2 + o(x^5)$$

EX $\begin{cases} f(x) = x + x^5 \\ f^2(x) = x^2 + 2x^6 + x^{10} \neq x^2 + o(x^8) \end{cases}$

14) • Se $f(x) = x + o(x^4)$, allora $f^2(x) = x^2 + o(x^5)$. V

15) • Se $f^2(x) = x^4 + o(x^8)$, allora $|f(x)| = x^2 + o(x^8)$. F

$$f^2(x) = x^4 + o(x^8) \qquad f^2(x) - x^4 = x^8 \cdot \omega(x)$$

$$f^2(x) = x^4 + x^8 \cdot \omega(x) \qquad |f(x)| = \sqrt{f^2(x)} = x^2 \cdot \sqrt{1+\omega(x)}$$

$$|f(x)| = x^2 + x^2\left(\sqrt{1+\omega(x)} - 1\right) = x^2 + x^2\omega_1(x)$$

$$\rightsquigarrow |f(x)| = x^2 + o(x^2)$$

EX $f(x) = x^2 + x^8 \qquad f^2(x) = x^4 + o(x^8)$

$$|f(x)| = x^2 + x^8 \quad \pm \quad x^2 + o(x^8)$$

16) • Se $f^2(x) = x^4 + o(x^8)$, allora $|f(x)| = x^2 + o(x^6)$. V

$$f(x) = x^2 + x^{6+\alpha} + o(x^{6+\alpha}) \quad \alpha > 0 \iff f^2(x) = x^4 + o(x^8)$$

$$|f(x)| = x^2 \pm x^6 \overbrace{|x^\alpha|}^{\omega(x)} = x^2 + o(x^6)$$

17) • Se $f(x) = x + o(x)$, allora $f(f(x)) = x + o(x)$. V

$$f(x) = x + x \cdot \omega(x)$$

$$f(f(x)) = x + x \cdot \omega(x) + [x + x \cdot \omega(x)]\, \omega[x + x \cdot \omega(x)] =$$

$$= x + x \{ \omega(x) + (1 + \omega(x)) \cdot \omega_1(x) \} = x + x \cdot \omega_2(x)$$

18) • Se $f(x) = x^3 + o(x^5)$, allora $f(f(x)) = x^9 + o(x^{11})$. V

$$f(x) = x^3 + x^5 \cdot \omega(x)$$

$$f(f(x)) = [x^3 + x^5 \cdot \omega(x)]^3 + [x^3 + x^5 \cdot \omega(x)]^5 \cdot \omega[x^3 + x^5 \cdot \omega(x)] =$$

$$= x^9 + 3x^{11}\omega(x) + 3x^{13}\omega^2(x) + x^{15} \cdot \omega^3(x) + x^{15} \cdot \omega_1(x) =$$

$$= x^9 + x^{11} \cdot \omega_2(x)$$

19) • Se $f(x) = x^2 + o(f(x))$, allora $f(\cancel{f}(x)) = x^2 + o(x^2)$. V

$$f(x) = x^2 + f(x) \cdot \omega(x)$$

$$f(x) = \frac{x^2}{1 - \omega(x)} = x^2 - x^2 + \frac{x^2}{1 - \omega(x)} =$$

$$= x^2 + x^2 \left(\frac{1}{1 - \omega(x)} - 1 \right) = x^2 + x^2 \overbrace{\left(\frac{\omega(x)}{1 - \omega(x)} \right)}^{\omega_1(x)} = x^2 + o(x^2)$$