BY GIMUSI 06/01/15

Sviluppi di Taylor 1

Argomenti: sviluppi di Taylor con centro nell'origine **Difficoltà:** ⋆⋆⋆

Prerequisiti: sviluppi di Taylor, operazioni algebriche con i polinomi di Taylor

Stabilire se le seguenti proposizioni sono vere o false (gli o piccolo si intendono per $x \to 0^+$).

Proposizione	V/F	Proposizione	V/F
$\arctan x = x - \frac{x^3}{3} + o(x^3)$	V	$\arctan x = x - \frac{x^3}{3} + o(x^4)$	V
$\arctan x = x - \frac{x^3}{3} + o(x^5)$	F	$\arctan x = x - \frac{x^3}{3} + o(x^2)$	V
$\arctan x = x - \frac{x^3}{7} + o(x^2)$	V	$\arctan x = x - \frac{x^3}{7} + o(x)$	V

In ogni riga delle seguenti tabelle sono indicati una funzione $f(x)$ ed un intero positivo n. Si chiede di determinare il polinomio di Taylor $P_n(x)$ (di grado minore o uguale a n) di $f(x)$ con centro nell'origine, ed il più grande intero k per cui vale lo sviluppo $f(x) = P_n(x) + o(x^k)$ per $x \to 0$.

	a) $f(x)$	n	$P_n(x)$	k	b) $f(x)$	n	$P_n(x)$	k
1)	$x^2(x^3+1)^4$	10	$x^2+4x^5+6x^8$	10	$\cos(2x)$	5	$1-2x^2+\frac{2}{3}x^4$	5
2)	$\cos^2 x$	5	$1-x^2+\frac{1}{3}x^4$	5	$\cos(x^2)$	8	$1-\frac{x^4}{2}+\frac{1}{24}x^8$	11
3)	$\log(1+3x^5)$	12	$3x^5-(9/2)x^{10}$	14	$\sin^3 x$	6	$x^3-(1/2)x^5$	6
4)	$\sqrt[3]{1-x}$	3	$1-\frac{x}{3}-\frac{1}{3}x^2-\frac{5}{81}x^3$	3	$\tan x$	5	$x+\frac{1}{3}x^3+\frac{2}{15}x^5$	6
5)	$\sqrt{1-2x^2}$	5	$1-x^2-(1/4)x^4$	5	$\cos^{20} x$	4	$1-10x^2+(155/3)x^4$	5

	$f(x)$	n	$P_n(x)$	k
6)	$x^7\sin(x^5)$	30	$x^{12}-(1/6)x^{22}$	31
7)	$e^x\sin(2x)$	4	$2x+2x^2-(1/3)x^3-x^4$	4
8)	$\arctan(x^2)\cdot\sin(x^3)$	9	$x^5-(1/3)x^9$	10
9)	$\arctan(x+x^3)$	4	$x+(2/3)x^3$	4
10)	$\log(\cos x+\sin x)$	4	$x-x^2+(2/3)x^3-(2/3)x^4$	4
11)	$\sin(\arctan x)$	5	$x-x^3/2+3x^5/8$	6
12)	$(1+\sin x)^x$	4	$1+x^2-x^3/2+2x^4/3$	4
13)	$\arcsin x$	5	$x+x^3/6+3x^5/40$	6

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Stabilire se le seguenti proposizioni sono vere o false (gli o piccolo si intendono per $x \to 0^+$).

Proposizione	V/F	Proposizione	V/F
$\arctan x = x - \frac{x^3}{3} + o(x^3)$	V	$\arctan x = x - \frac{x^3}{3} + o(x^4)$	V
$\arctan x = x - \frac{x^3}{3} + o(x^5)$	F	$\arctan x = x - \frac{x^3}{3} + o(x^2)$	V
$\arctan x = x - \frac{x^3}{7} + o(x^2)$	V	$\arctan x = x - \frac{x^3}{7} + o(x)$	V

$$\text{ARCTAN}X = X - \frac{X^3}{3} + \frac{X^5}{5} + \dots$$

$\text{ARCTAN}X = X - \frac{X^3}{3} + o(X^3)$ V ⇝ FORM. DI TAYLOR PER $n = 3$

$\text{ARCTAN}X = X - \frac{X^3}{3} + o(X^4)$ V ⇝ FORM. DI TAYLOR PER $n = 4$

$\text{ARCTAN}X = X - \frac{X^3}{3} + o(X^5)$ F ⇝ MANCA IL TERMINE $X^5/5$

$\text{ARCTAN}X = X - \frac{X^3}{3} + o(X^2)$ V ⇝ FORM. DI TAYLOR PER $n = 2$

$o(X^2)$ SI "MANGIA" $X^3/3$

$$= \text{ARCTAN}X - X = X^2\left(-\frac{X}{3} + \omega(X)\right) \quad \to 0$$

$\text{ARCTAN}X = X - \frac{X^3}{7} + o(X^2)$ V ⇝ FORM. DI TAYLOR PER $n = 2$

$o(X^2)$ SI "MANGIA" $X^3/7$

$$= \text{ARCTAN}X - X = X^2\left(-\frac{X}{7} + \omega(X)\right) \quad \to 0$$

$\text{ARCTAN}X = X - \frac{X^3}{7} + o(X)$ V ⇝ FORM. DI TAYLOR PER $n = 1$

$o(X)$ SI "MANGIA" $X^3/3$

$$= \text{ARCTAN}X - X = X\left(-\frac{X^2}{7} + \omega(X)\right) \quad \to 0$$

In ogni riga delle seguenti tabelle sono indicati una funzione $f(x)$ ed un intero positivo n. Si chiede di determinare il polinomio di Taylor $P_n(x)$ (di grado minore o uguale a n) di $f(x)$ con centro nell'origine, ed il più grande intero k per cui vale lo sviluppo $f(x) = P_n(x) + o(x^k)$ per $x \to 0$.

1.a) $x^2(x^3+1)^5 \qquad n=10$

1.a) $x^2(x^3+1)^5 = x^2(1+5x^3+6x^6+5x^9+x^{12})$

$P_{10}(x) = x^2+5x^5+6x^8 + o(x^{10}) \qquad k=10$

1.b) $\cos(2x) \quad n=5$

$\cos\delta = 1 - \frac{\delta^2}{2!} + \frac{\delta^5}{5!} - \frac{\delta^6}{6!} + o(\delta^6) \qquad \delta = 2x \qquad \delta \to 0$

$\cos(2x) = 1 - \frac{2^2}{2!}x^2 + \frac{2^5}{5!}x^5 - \frac{2^6}{6!}x^6 + o(x^6) =$

$= 1 - 2x^2 + \frac{16}{25}x^5 - \frac{65}{720}x^6 + o(x^6)$

$P_5 = 1 - 2x^2 + \frac{2}{3}x^5 \qquad k=5$

2.a) $\cos^2 x \quad n=5$

$\cos\delta = 1 - \frac{\delta^2}{2!} + \frac{\delta^5}{5!} - \frac{\delta^6}{6!} + o(\delta^6) \qquad \delta \to 0$

$(\cos x)^2 = 1 - 2\frac{x^2}{2!} + \left(\frac{x^2}{2!}\right)^2 + 2\frac{x^5}{5!} - 2\frac{x^6}{6!} - 2\frac{\delta^6}{2!5!} + o(x^6) =$

$= 1 - x^2 + \left(\frac{1}{5} + \frac{1}{12}\right)x^5 + o(x^5)$ 1/3

$P_5(x) = 1 - x^2 + \frac{1}{3}x^5 \qquad k=5$

2.b) $\cos(x^2) \qquad n=8 \qquad \cos\delta = 1 - \frac{\delta^2}{2!} + \frac{\delta^5}{5!} - \frac{\delta^6}{6!} + o(\delta^6) \quad \delta \to 0$

$\cos(x^2) = 1 - \frac{x^5}{2} + \frac{x^8}{5!} - \frac{x^{12}}{6!} + o(x^{12})$

$P_8(x) = 1 - \frac{x^5}{2} + \frac{1}{25}x^8 \qquad k=11$

3.a) $\log(1+3x^5)$ $\quad n=12$

$$\log(1+\delta) = \delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} - \frac{\delta^5}{5} + \dots \qquad \delta \to 0$$

$$\log(1+3x^5) = 3x^5 - \frac{1}{2}(3x^5)^2 + \frac{1}{3}(3x^5)^3 + o(x^{15}) =$$

$$= 3x^5 - \frac{9}{2}x^{10} + 9x^{15} + o(x^{15})$$

$$P_{12}(x) = 3x^5 - \frac{9}{2}x^{10} \qquad K = 15$$

3.b) $\sin^3 x$ $\quad n=6$

$$\sin\delta = \delta - \frac{\delta^3}{6} + \frac{\delta^5}{120} - \frac{\delta^7}{7!} + o(\delta^7) \qquad \delta \to 0$$

$$(\sin x)^3 = x^3 - 3x^2\frac{x^3}{6} + 3x\left(\frac{x^3}{6}\right)^2 + 3x^2\frac{x^5}{120} - \frac{x^7}{7!} + o(x^7) =$$

$$= x^3 - \frac{1}{2}x^5 + \left(\frac{1}{2} + \frac{1}{50} - \frac{1}{7!}\right)x^7 + o(x^7)$$

$$P_6(x) = x^3 - \frac{1}{2}x^5 \qquad n = 6$$

5.a) $\sqrt[3]{1-x}$ $\quad n=3$

$$(1+\delta)^{1/3} = 1 + \frac{1}{3}\delta + \frac{1}{2}\frac{1}{3}\left(-\frac{2}{3}\right)\delta^2 + \frac{1}{6}\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\delta^3 + o(\delta^3)$$

$$(1-x)^{1/3} = 1 - \frac{x}{3} - \frac{1}{9}x^2 - \frac{5}{81}x^3 + o(x^3)$$

$$P_3(x) = 1 - \frac{x}{3} - \frac{1}{9}x^2 - \frac{5}{81}x^3 \qquad K = 3$$

5.b) $\tan x$ $\quad n=5$

$$\tan(0) = 0 \qquad (\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x \qquad \tan'(0) = 1$$

NOTAZIONE: $\tan x = T \qquad T' = 1 + T^2$

$T'' = 2TT' = 2T + 2T^3 \qquad TAN''(0) = 0$

$T''' = 2T' + 6T^2T' = 2(1+T^2)(1+3T^2) =$

$= 2 + 8T^2 + 6T^4 \qquad TAN'''(0) = 2$

$T^{IV} = 16TT' + 24T^3T' = 16T + 16T^3 + 24T^3 + 24T^5 =$

$= 16T + 40T^3 + 24T^5 \qquad TAN^{IV}(0) = 0$

$T^{V} = 16T' + 120T^2T' + 120T^4T' \qquad TAN^{V}(0) = 16$

$TAN(x) = x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + o(x^5) =$

$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^5)$

$P_5(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \qquad K = 6$

5.a) $\sqrt{1-2x^2} \qquad n = 5$

$(1+\delta)^{1/2} = 1 + \frac{1}{2}\delta + \frac{1}{2}\,\frac{1}{2}\left(-\frac{1}{2}\right)\delta^2 + \frac{1}{6}\,\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\delta^3 + o(\delta^3) =$

$= 1 + \frac{1}{2}\delta - \frac{1}{8}\delta^2 + \frac{1}{16}\delta^3 + o(\delta^3) \qquad \delta \to 0$

$(1-2x^2)^{1/2} = 1 + \frac{1}{2}(-2x^2) - \frac{1}{8}(-2x^2)^2 + \frac{1}{16}(-2x^2)^3 + o(x^6) =$

$= 1 - x^2 - \frac{1}{4}x^4 - \frac{1}{2}x^6 + o(x^6)$

$P_5(x) = 1 - x^2 - \frac{1}{4}x^4 \qquad n = 5$

5.b) $\cos^{20} x \qquad n = 4$

$\cos\delta = 1 - \frac{1}{2}\delta^2 + \frac{1}{24}\delta^4 - \frac{1}{6!}\delta^6 + o(\delta^6)$

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$(\cos\delta)^{20} = 1 - \frac{20}{2}\delta^2 + \binom{20}{2}\left(-\frac{1}{2}\delta^2\right)^2 + \frac{20}{24}\delta^4 + o(\delta^6) =$

$$= 1 - 10\delta^2 + \frac{95}{2}\delta^5 + \frac{5}{6}\delta^5 + o(\delta^6) =$$

$$\frac{95}{2} + \frac{5}{6} = \frac{285+5}{6} = \frac{155}{3}$$

$$= 1 - 10\delta^2 + \frac{155}{3}\delta^5 + o(\delta^6)$$

$$\cos^{20} x = 1 - 10x^2 + \frac{155}{3}x^5 + o(x^6) \qquad n = 5$$

6) $x^7 \sin(x^5) \quad m = 30$

$$\sin\delta = \delta - \frac{1}{3!}\delta^3 + \frac{1}{5!}\delta^5 + o(\delta^5)$$

$$\sin(x^5) = x^5 - \frac{1}{3!}x^{15} + \frac{1}{5!}x^{25} + o(x^{25})$$

$$x^7\sin(x^5) = x^{12} - \frac{1}{3!}x^{22} + \frac{1}{5!}x^{32} + o(x^{32})$$

$$P_{30}(x) = x^{12} - \frac{1}{6}x^{22} \qquad n = 31$$

7) $e^x \sin(2x) \quad m = 5$

$$\sin\delta = \delta - \frac{1}{3!}\delta^3 + \frac{1}{5!}\delta^5 + o(\delta^5)$$

$$\sin(2x) = 2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 + o(x^5) =$$

$$= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + o(x^5)$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + o(x^4)$$

$$e^x\sin(2x) = 2x - \frac{4}{3}x^3 + x\left(2x - \frac{4}{3}x^3\right) + \frac{1}{2}x^2(2x) +$$

$$+ \frac{1}{6}x^3(2x) + o(x^4) =$$

$$= 2x - \frac{5}{3}x^3 + 2x^2 - \frac{5}{3}x^5 + x^3 + \frac{1}{3}x^5 + o(x^5) =$$

$$= 2x + 2x^2 - \frac{1}{3}x^3 - x^5 + o(x^5)$$

$$P_5(x) = 2x + 2x^2 - \frac{1}{3}x^3 - x^5 \qquad k = 5$$

8) $\arctan(x^2) \cdot \sin(x^3) \qquad n = 9$

$$\arctan(\delta) = \delta - \frac{1}{3}\delta^3 + \frac{1}{5}\delta^5 + o(\delta^5)$$

$$\arctan(x^2) = x^2 - \frac{1}{3}x^6 + \frac{1}{5}x^{10} + o(x^{10})$$

$$\sin\delta = \delta - \frac{1}{3!}\delta^3 + \frac{1}{5!}\delta^5 + o(\delta^5)$$

$$\sin x^3 = x^3 - \frac{1}{6}x^9 + \frac{1}{5!}x^{15} + o(x^{15})$$

$$\arctan(x^2) \cdot \sin(x^3) = x^3\left(x^2 - \frac{1}{3}x^6\right) - \frac{1}{6}x^9 \cdot x^2 + o(x^{11}) =$$

$$= x^5 - \frac{1}{3}x^9 - \frac{1}{6}x^{11} + o(x^{11})$$

$$P_9(x) = x^5 - \frac{1}{3}x^9 \qquad n = 10$$

9) $\arctan(x + x^3) \qquad n = 5$

$$\arctan(\delta) = \delta - \frac{1}{3}\delta^3 + \frac{1}{5}\delta^5 + o(\delta^5)$$

$$\arctan(x + x^3) = (x + x^3) - \frac{1}{3}(x + x^3)^3 + \frac{1}{5}x^5 + o(x^5) =$$

$$= x + x^3 - \frac{1}{3}x^3 - x^5 + \frac{1}{5}x^5 + o(x^5) =$$

$$= x + \frac{2}{3}x^3 - \frac{4}{5}x^5 + o(x^5)$$

$$P_4(x) = x + \frac{2}{3}x^3 \qquad n = 4$$

10) $\log(\cos x + \sin x) \quad n = 4$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4) \qquad \sin x = x - \frac{x^3}{6} + o(x^4)$$

$$\cos x + \sin x = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)$$

$$\log(1+\delta) = \delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} - \frac{\delta^4}{4} + o(\delta^4)$$

$$\log(\cos x + \sin x) = \log\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)\right) =$$

$$= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{1}{2}\left(x - \frac{x^2}{2} - \frac{x^3}{6}\right)^2 + \frac{1}{3}\left(x - \frac{x^2}{2}\right)^3 - \frac{1}{4}x^4 + o(x^4) =$$

$$= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{1}{2}\left(x^2 + \frac{x^4}{4} - x^3 - \frac{x^4}{3}\right) + \frac{1}{3}\left(x^3 - \frac{3}{2}x^4\right) - \frac{x^4}{4} + o(x^4) =$$

$$= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^3}{3} - \frac{x^4}{2} - \frac{x^4}{4} + o(x^4) =$$

$$= x - x^2 + \left(-\frac{1}{6} + \frac{1}{2} + \frac{1}{3}\right)x^3 + \left(\frac{1}{24} - \frac{1}{8} + \frac{1}{6} - \frac{1}{2} - \frac{1}{4}\right)x^4 + o(x^4) =$$

$$\frac{-1+3+2}{6} = \frac{2}{3} \qquad \frac{1-3+4-12-6}{24} = -\frac{2}{3}$$

$$= x - x^2 + \frac{2}{3}x^3 - \frac{2}{3}x^4 + o(x^4)$$

$$P_4(x) = x - x^2 + \frac{2}{3}x^3 - \frac{2}{3}x^4 \qquad n = 4$$

11) $\sin(\arctan x) \quad n = 5$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + o(x^5)$$

$$\sin\delta = \delta - \frac{\delta^3}{6} + \frac{\delta^5}{120} + o(\delta^5)$$

$$\sin(\arctan x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{6}\left(x - \frac{1}{3}x^3\right)^3 + \frac{x^5}{120} + o(x^5) =$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{6}\left(x^3 - x^5\right) + \frac{x^5}{120} + o(x^5) =$$

$$= x - \frac{1}{2}x^3 + \frac{3}{8}x^5 + o(x^5)$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{120} = \frac{24+20+1}{120} = \frac{45}{120} = \frac{3}{8}$$

$$P_5(x) = x - \frac{1}{2}x^3 + \frac{3}{8}x^5 \qquad n = 6$$

12) $(1+\sin x)^x \quad n=5$

$$(1+\sin x)^x = e^{x \log(1+\sin x)}$$

$$\sin x = x - \frac{x^3}{6} + o(x^5)$$

$$\log(1+\sin x) = \log\left(1+x-\frac{x^3}{6}+o(x^5)\right) =$$

$$= x - \frac{x^3}{6} - \frac{1}{2}\left(x-\frac{x^3}{6}\right)^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5 + o(x^5) =$$

$$= x - \frac{x^3}{6} - \frac{1}{2}\left(x^2 - \frac{x^5}{3}\right) + \frac{1}{3}x^3 - \frac{1}{5}x^5 + o(x^5) =$$

$$= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^5 + o(x^5)$$

$$x\log(1+\sin x) = x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^5 + o(x^5)$$

$$(1+\sin x)^x = e^{x\log(1+\sin x)} = 1 + x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^5 +$$

$$+ \frac{1}{2}\left(x^2\right)^2 + o(x^5) = 1 + x^2 - \frac{1}{2}x^3 + \frac{2}{3}x^5 + o(x^5)$$

$$P_5(x) = 1 + x^2 - \frac{1}{2}x^3 + \frac{2}{3}x^5 \qquad n=5$$

13) $\arcsin x \quad n=5$

$$\arcsin(0) = 0$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad \arcsin'(0) = 1$$

$$(\arcsin x)'' = -\frac{1}{2}\,\frac{-2x}{\sqrt{(1-x^2)^3}} \qquad \arcsin''(0) = 0$$

$$(\arcsin x)''' = \left(x(1-x^2)^{-3/2}\right)' =$$

$$= (1-x^2)^{-3/2} + x\left[-\frac{3}{2}(1-x^2)^{-5/2}\cdot(-2x)\right] =$$

$$= (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2} \qquad \arcsin'''(0) = 1$$

$$\arcsin^{IV} x = -\frac{3}{2}(1-x^2)^{-5/2}(-2x) + 6x(1-x^2)^{-5/2} + 3x^2\left(-\frac{5}{2}\right)(1-x^2)^{-7/2}(-2x) =$$

$$= 9x(1-x^2)^{-5/2} + 15x^3(1-x^2)^{-7/2} \qquad \arcsin^{IV}(0) = 0$$

$$\arcsin^{V} x = 9(1-x^2)^{-5/2} + 9x\left(-\frac{5}{2}\right)(1-x^2)^{-7/2}(-2x)\ldots \qquad \arcsin^{V}(0) = 9$$

$$\arcsin x = x + \frac{x^3}{6} + \frac{9}{5!}x^5 + o(x^5) =$$

$$= x + \frac{x^3}{6} + \frac{3}{40}x^5 + o(x^5)$$

$$P_5(x) = x + \frac{x^3}{6} + \frac{3}{40}x^5 \qquad n = 6$$

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