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Induzione 1

Argomenti: principio di induzione **Difficoltà**: $\star\star$

Prerequisiti: principio di induzione

1. Dimostrare le seguenti identità
$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=0}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

2. Dimostrare che per ogni $a \neq 1$ si ha che
$$1 + a + a^2 + \ldots + a^n = \sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}.$$
Cosa succede nel caso $a = 1$? Estendere il risultato alla somma a segni alterni (con ultimo esponente pari)
$$1 - a + a^2 - a^3 + \ldots + a^{2n}.$$

3. Dimostrare che per ogni intero $n \geq 1$ si ha che
$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

4. Quantificare e dimostrare le *identità trigonometriche di Lagrange*:
$$\sum_{k=0}^{n} \sin(kx) = \frac{\cos(x/2) - \cos[(n+1/2)x]}{2\sin(x/2)}, \qquad \sum_{k=0}^{n} \cos(kx) = \frac{1 + \sin[(n+1/2)x]}{2\sin(x/2)}.$$

5. Sia $f : [0, +\infty) \to [0, +\infty)$ la funzione definita da
$$f(x) = \frac{x}{x+1}.$$
Determinare una formula esplicita per la funzione ottenuta componendo f con se stessa n volte.

6. Una successione è stata definita ponendo $a_0 = 3$, $a_1 = a_0^2$, $a_2 = a_1^2$, $a_3 = a_2^2$, e così via. Determinare a_{2014}.

7. Determinare per quali valori di $n \in \mathbb{N}$ valgono le seguenti disuguaglianze
$$2^n \geq n, \qquad 2^n \geq n^3, \qquad n! \geq 2^n, \qquad n! \leq 2^{n^2}, \qquad 3^n + 4^n < 5^n.$$

8. Dimostrare che $(2n)! \geq 2^n (n!)^2$ per ogni intero $n \geq 37$.

9. Dimostrare che per ogni intero $n \geq 1$ valgono le seguenti disuguaglianze con i binomiali (pensare a quale relazione c'è tra queste disuguaglianze):
$$\binom{2n}{n} \leq 4^n, \qquad \binom{2n}{n} \leq \frac{4^n}{\sqrt{3n+1}}, \qquad \binom{2n}{n} \leq \frac{4^n}{\sqrt{3n}}.$$

1. Dimostrare le seguenti identità

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=0}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

1.a) $$\sum_{n=0}^{m} K = \frac{m(m+1)}{2}$$

PASSO BASE : $m=0 \qquad \sum_{n=0}^{0} K = 0$

PASSO INDUTTIVO : $$\sum_{n=0}^{m} K = \frac{m(m+1)}{2} \Rightarrow \sum_{n=0}^{m+1} K = \frac{(m+1)(m+2)}{2}$$

DIM : $$\sum_{n=0}^{m+1} K = \sum_{n=0}^{m} K + (m+1) = \frac{m(m+1)}{2} + (m+1) =$$

$$= \frac{m(m+1)+2(m+1)}{2} = \frac{(m+1)(m+2)}{2}$$

1.b) $$\sum_{K=0}^{m} K^2 = \frac{m(m+1)(2m+1)}{6}$$

PASSO BASE : $m=0 \qquad \sum_{n=0}^{0} K^2 = 0$

PASSO INDUTTIVO :

$$\sum_{K=0}^{m} K^2 = \frac{m(m+1)(2m+1)}{6} \Rightarrow \sum_{K=0}^{m+1} K^2 = \frac{(m+1)(m+2)(2m+3)}{6}$$

DIM : $$\sum_{K=0}^{m+1} K^2 = \sum_{K=0}^{m} K^2 + (m+1)^2 = \frac{m(m+1)(2m+1)}{6} + (m+1)^2 =$$

$$= \frac{m(m+1)(2m+1)+6(m+1)^2}{6} = \frac{(m+1)(2m^2+m+6m+6)}{6} =$$

$$= \frac{(m+1)(2m^2+7m+6)}{6} = \frac{(m+1)(m+2)(2m+3)}{6}$$

$$2m^2+7m+6=0 \qquad m = \frac{-7\pm\sqrt{49-48}}{4} = \begin{cases} -2 \\ -3/2 \end{cases}$$

$$2m^2+7m+6 = 2(m+2)(m+3/2) = (m+2)(2m+3)$$

1.c) $\sum_{n=0}^{m} k^3 = \left(\frac{m(m+1)}{2}\right)^2$

PASSO BASE: $m=0$ $\quad \sum_{n=0}^{0} k^3 = 0$

PASSO INDUTTIVO: $\sum_{n=0}^{m} k^3 = \left(\frac{m(m+1)}{2}\right)^2 \Rightarrow \sum_{n=0}^{m+1} k^3 = \left(\frac{(m+1)(m+2)}{2}\right)^2$

DIM: $\sum_{n=0}^{m+1} k^3 = \sum_{n=0}^{m} k^3 + (m+1)^3 = \left(\frac{m(m+1)}{2}\right)^2 + (m+1)^3 =$

$$= \frac{(m+1)^2}{4}\left(m^2+4m+4\right) = \frac{(m+1)^2(m+2)^2}{4} = \left(\frac{(m+1)(m+2)}{2}\right)^2$$

2. Dimostrare che per ogni $a \neq 1$ si ha che

$$1 + a + a^2 + \ldots + a^n = \sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}.$$

PASSO BASE: $m=0$ $\quad 1 = \frac{a^{0+1}-1}{a-1} = 1$

PASSO INDUTTIVO: $\sum_{n=0}^{m} a^k = \frac{a^{m+1}-1}{a-1} \Rightarrow \sum_{n=0}^{m+1} a^k = \frac{a^{m+2}-1}{a-1}$

DIM: $\sum_{n=0}^{m+1} a^k = \sum_{n=0}^{m} a^k + a^{m+1} = \frac{a^{m+1}-1}{a-1} + a^{m+1} =$

$$= \frac{\cancel{a^{m+1}} - 1 + a^{m+2} - \cancel{a^{m+1}}}{a-1} = \frac{a^{m+2}-1}{a-1}$$

Cosa succede nel caso $a = 1$? Estendere il risultato alla somma a segni alterni (con ultimo esponente pari)

$$1 - a + a^2 - a^3 + \ldots + a^{2n}.$$

$\underline{a=1}$ $\quad \sum_{n=0}^{m} a^k = \sum_{n=0}^{m} 1 = m+1$

$$\sum_{n=0}^{2m} (-1)^k a^k = \sum_{n=0}^{m} a^{2k} - \sum_{k=1}^{m} a^{2n-1} = \sum_{n=0}^{m} (a^2)^k - \frac{1}{a}\left(\sum_{n=0}^{m} (a^2)^k - 1\right) =$$

$$= \sum_{n=0}^{m} (a^2)^k \left(1 - \frac{1}{a}\right) + \frac{1}{a} = \frac{(a^2)^{(m+1)}-1}{a^{\cancel{2}}+1}\ \frac{\cancel{a-1}}{a} + \frac{1}{a} = \frac{a^{2m+\cancel{3}\,1} \cancel{-1} + \cancel{a} \cancel{+1}}{\cancel{a}(a+1)} = \frac{a^{2m+1}+1}{a+1}$$

3. Dimostrare che per ogni intero $n \geq 1$ si ha che

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

PASSO BASE: $\quad m = 1 \quad \sum_{n=1}^{1} \frac{1}{k(n+1)} = \frac{1}{2}$

PASSO INDUTTIVO: $\quad \sum_{n=1}^{m} \frac{1}{n(n+1)} = \frac{m}{m+1} \Rightarrow \sum_{n=1}^{m+1} \frac{1}{n(n+1)} = \frac{m+1}{m+2}$

DIM: $\quad \sum_{n=1}^{m+1} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \frac{1}{n(n+1)} + \frac{1}{(m+1)(m+2)} = \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} =$

$$= \frac{m(m+2)+1}{(m+1)(m+2)} = \frac{m^2+2m+1}{(m+1)(m+2)} = \frac{(m+1)^{\cancel{2}}}{\cancel{(m+1)}(m+2)} = \frac{m+1}{m+2}$$

4. Quantificare e dimostrare le *identità trigonometriche di Lagrange*:

$$\sum_{k=0}^{n} \sin(kx) = \frac{\cos(x/2) - \cos[(n+1/2)x]}{2\sin(x/2)}, \qquad \sum_{k=0}^{n} \cos(kx) = \frac{\overset{\sin(x/2)}{\cancel{1}} + \sin[(n+1/2)x]}{2\sin(x/2)}.$$

4.a) $\quad \sum_{n=0}^{m} \sin(nx) = \frac{\cos(x/2) - \cos[(m+1/2)x]}{2\sin(x/2)} \qquad \forall x \in \mathbb{R} - \{2k\pi\} \quad k \in \mathbb{Z}$

PASSO BASE: $m = 0 \quad \sum_{n=0}^{0} \sin(nx) = \frac{\cos(x/2) - \cos(x/2)}{2\sin(x/2)} = 0$

PASSO INDUTTIVO: $\quad \sum_{n=0}^{m} \sin(nx) = \frac{\cos(x/2) - \cos[(m+1/2)x]}{2\sin(x/2)}$

$$\Rightarrow \sum_{n=0}^{m+1} \sin(nx) = \frac{\cos(x/2) - \cos[(m+3/2)x]}{2\sin(x/2)}$$

DIM: $\quad \sum_{n=0}^{m+1} \sin(nx) = \sum_{n=0}^{m} \sin(nx) + \sin[(m+1)x] =$

$$= \frac{\cos(x/2) - \cos[(m+1/2)x]}{2\sin(x/2)} + \sin[(m+1)x] =$$

$$= \frac{\cos(x/2) - \cos[(m+1/2)x] + 2\sin[(m+1)x]\sin(x/2)}{2\sin(x/2)} =$$

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$$\begin{cases} 2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B) \\ 2\sin[(m+1)x]\,\sin(x/2) = \cos[(m+1/2)x] - \cos[(m+3/2)x] \end{cases}$$

$$= \frac{\cos(x/2) - \cancel{\cos[(m+1/2)x]} + \cancel{\cos[(m+1/2)x]} - \cos[(m+3/2)x]}{2\sin(x/2)} =$$

$$= \frac{\cos(x/2) - \cos[(m+3/2)x]}{2\sin(x/2)}$$

5.b) $$\sum_{n=0}^{m} \cos(nx) = \frac{\overset{\sin(x/2)}{\cancel{1}} + \sin[(m+1/2)x]}{2\sin(x/2)} \qquad \forall x \in \mathbb{R} - \{2k\pi\} \quad k \in \mathbb{Z}$$

PASSO BASE: $m=0$ $$\sum_{n=0}^{0} \cos(nx) = \frac{\sin(x/2) + \sin(x/2)}{2\sin(x/2)} = 1$$

PASSO INDUTTIVO: $$\sum_{n=0}^{m} \cos(nx) = \frac{\sin(x/2) + \sin[(m+1/2)x]}{2\sin(x/2)}$$

$$\Rightarrow \sum_{n=0}^{m+1} \cos(nx) = \frac{\sin(x/2) + \sin[(m+3/2)x]}{2\sin(x/2)}$$

DIM: $$\sum_{n=0}^{m+1} \cos(nx) = \sum_{n=0}^{m} \cos(nx) + \cos[(m+1)x] =$$

$$= \frac{\sin(x/2) + \sin[(m+1/2)x]}{2\sin(x/2)} + \cos[(m+1)x] =$$

$$= \frac{\sin(x/2) + \sin[(m+1/2)x] + 2\sin(x/2)\cos[(m+1)x]}{2\sin(x/2)} =$$

$$\begin{cases} 2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B) \\ 2\sin(x/2)\cos[(m+1)x] = \sin[(m+3/2)x] + \sin[-(m+1/2)x] \end{cases}$$

$$= \frac{\sin(x/2) + \cancel{\sin[(m+1/2)x]} + \sin[(m+3/2)x] - \cancel{\sin[(m+1/2)x]}}{2\sin(x/2)} =$$

$$= \frac{\sin(x/2) + \sin[(m+3/2)x]}{2\sin(x/2)}$$

5. Sia $f : [0, +\infty) \to [0, +\infty)$ la funzione definita da

$$f(x) = \frac{x}{x+1}.$$

Determinare una formula esplicita per la funzione ottenuta componendo f con se stessa n volte.

$$f_2 = f \circ f = \frac{\frac{x}{x+1}}{\frac{x}{x+1} + 1} = \frac{\frac{x}{x+1}}{\frac{2x+1}{x+1}} = \frac{x}{2x+1} \rightsquigarrow f_n = \frac{x}{nx+1} \ ?$$

PASSO BASE: $n = 2$

PASSO INDUTTIVO: $f_n = \frac{x}{nx+1} \Rightarrow f_{n+1} = \frac{x}{(n+1)x + 1}$

DIM: $f_{n+1} = f \circ f_n = \frac{\frac{x}{nx+1}}{\frac{x}{nx+1} + 1} = \frac{\frac{x}{nx+1}}{\frac{x+nx+1}{nx+1}} = \frac{x}{(n+1)x+1}$

6. Una successione è stata definita ponendo $a_0 = 3$, $a_1 = a_0^2$, $a_2 = a_1^2$, $a_3 = a_2^2$, e così via. Determinare a_{2014}.

$a_0 = 3 \quad a_1 = a_0^2 = 9 \quad a_2 = a_1^2 = a_0^4 = 81 \quad a_3 = a_2^2 = a_0^8 = 6561 \quad a_n = 3^{2^n}$?

PASSO BASE: $a_0 = 3$

PASSO INDUTTIVO: $a_n = 3^{2^n} \Rightarrow a_{n+1} = 3^{2^{n+1}}$

DIM: $a_{n+1} = (a_n)^2 = 3^{2^n} \cdot 3^{2^n} = 3^{2 \cdot 2^n} = 3^{2^{n+1}} \rightsquigarrow a_{2014} = 3^{2^{2014}}$

7. Determinare per quali valori di $n \in \mathbb{N}$ valgono le seguenti disuguaglianze

$$2^n \geq n, \qquad 2^n \geq n^3, \qquad n! \geq 2^n, \qquad n! \leq 2^{n^2}, \qquad 3^n + 4^n < 5^n.$$

7.a) $2^n \geq n$

PASSO BASE: $n = 0 \quad 2^0 = 1 \geq 0$

PASSO INDUTTIVO: $2^n \geq n \Rightarrow 2^{n+1} \geq n+1$

DIM: $2^{n+1} = 2 \cdot 2^n \geq 2n \overset{?}{\geq} n+1$

$2n \geq n+1 \Rightarrow n \geq 1$

PASSO BASE BIS: $n=1 \quad 2^1 = 2 \geq 1$

$\rightsquigarrow 2^n \geq n \quad \forall n \in \mathbb{N}$

7.b) $2^n \geq n^3$

PASSO BASE: $n=0 \quad 2^0 = 1 \geq 0$

PASSO INDUTTIVO: $2^n \geq n^3 \Rightarrow 2^{n+1} \geq (n+1)^3$

DIM: $2^{n+1} = 2 \cdot 2^n \geq 2n^3 \overset{?}{\geq} (n+1)^3$

$2n^3 \geq (n+1)^3 \Rightarrow \sqrt[3]{2}\, n \geq n+1$ $\quad$ (x^3 MONOTONA)

$(\sqrt[3]{2}-1)n \geq 1 \qquad n \geq \dfrac{1}{\sqrt[3]{2}-1} \simeq 3.85$

PASSO BASE BIS: $n=4 \quad 2^4 = 16 \geq 4^3 = 64$ NO

$n=5 \quad 2^5 = 32 \geq 5^3 = 125$ NO

$n=6 \quad 2^6 = 64 \geq 6^3 = 216$ NO

$n=7 \quad 2^7 = 128 \geq 7^3 = 343$ NO

$n=8 \quad 2^8 = 256 \geq 8^3 = 512$ NO

$n=9 \quad 2^9 = 512 \geq 9^3 = 729$ NO

$n=10 \quad 2^{10} = 1024 \geq 10^3 = 1000$ SÍ

$\rightsquigarrow 2^n \geq n^3 \quad \forall n \geq 10$

7.c) $n! \geq 2^n$

PASSO BASE: $n=0 \quad 0! = 1 \geq 2^0 = 1$

PASSO INDUTTIVO: $n! \geq 2^n \Rightarrow (n+1)! \geq 2^{n+1}$

DIM: $(n+1)! = (n+1) \cdot n! \geq (n+1) \cdot 2^n \overset{?}{\geq} 2^{n+1} = 2 \cdot 2^n$

$$(n+1)\cdot 2^n \geq 2\cdot 2^n \Rightarrow n+1 \geq 2 \qquad n \geq 1$$

PASSO BASE BIS: $n = 1 \quad 1! \geq 2^1$ NO

$n = 2 \quad 2! = 2 \geq 2^2 = 4$ NO

$n = 3 \quad 3! = 6 \geq 2^3 = 8$ NO

$n = 4 \quad 4! = 24 \geq 2^4 = 16$ SÍ

$\rightsquigarrow n! \geq 2^n \quad n \geq 4$

7.d) $n! \leq 2^{n^2}$

PASSO BASE: $n = 0 \quad 0! = 1 \leq 2^0 = 1$

PASSO INDUTTIVO: $n! \leq 2^{n^2} \Rightarrow (n+1)! \leq 2^{(n+1)^2}$

DIM: $(n+1)! = (n+1)\, n! \leq (n+1)\, 2^{n^2} \overset{?}{\leq} 2^{(n+1)^2}$

$$(n+1)\, 2^{n^2} \leq 2^{(n+1)^2} = 2^{n^2} \cdot 2^{2n+1} \qquad (n+1) \leq 2^{2n+1}$$

$$n+1 \leq 2 \cdot 4^n \qquad 4^n \geq \frac{n+1}{2}$$

PASSO BASE: $n = 0 \quad 4^0 = 1 \geq 1/2$

PASSO INDUTTIVO: $4^n \geq \frac{n+1}{2} \Rightarrow 4^{n+1} \geq \frac{n+2}{2}$

DIM: $4^{n+1} = 4 \cdot 4^n \geq \overset{2}{\cancel{4}} \cdot \frac{n+1}{\cancel{2}} \overset{?}{\geq} \frac{n+2}{2}$

$$4(n+1) \geq n+2 \qquad 4n - n \geq -2 \qquad 3n \geq -2$$

$\rightsquigarrow 4^n \geq \frac{n+1}{2} \quad \forall n \in \mathbb{N}$

$\rightsquigarrow n! \leq 2^{n^2} \quad \forall n \in \mathbb{N}$

7.e) $3^n + 5^n < 5^n$

PASSO BASE: $n=3$ $\quad 3^3 + 5^3 = 27 + 65 = 92 < 5^3 = 125$

PASSO INDUTTIVO: $3^n + 5^n < 5^n \Rightarrow 3^{n+1} + 5^{n+1} < 5^{n+1}$

DIM: $5^{n+1} = 5 \cdot 5^n > 5 \cdot (3^n + 5^n) \overset{?}{>} 3^{n+1} + 5^{n+1}$

$$5 \cdot (3^n + 5^n) = 5 \cdot 3^n + 5 \cdot 5^n > 3^{n+1} + 5^{n+1} =$$
$$= 3 \cdot 3^n + 5 \cdot 5^n \quad \forall n \in \mathbb{N}$$

$\rightsquigarrow 3^n + 5^n < 5^n \quad \forall n \geq 3$

8. Dimostrare che $(2n)! \geq 2^n(n!)^2$ per ogni intero $n \geq 37$.

PASSO BASE: $n=0$ $\quad (2 \cdot 0)! = 1 \geq 1 \cdot (0!)^2 = 1$

PASSO INDUTTIVO: $(2n)! \geq 2^n(n!)^2 \Rightarrow (2n+2)! \geq 2^{n+1}[(n+1)!]^2$

DIM: $(2n+2)! = (2n)!(2n+2) \cdot (2n+1) \geq 2^n(n!)^2(2n+2)(2n+1) \overset{?}{\geq}$

$$\overset{?}{\geq} 2^{n+1}[(n+1)!]^2$$

$$\cancel{2^n}\cancel{(n!)^2}(\cancel{2n+2})(2n+1) \geq 2^{n+1}[(n+1)!]^2 =$$

$$= \cancel{2} \cdot \cancel{2^n}\cancel{(n!)^2}(n+1)^{\cancel{2}} \qquad 2n+1 \geq n+1 \qquad \forall n \in \mathbb{N}$$

$\rightsquigarrow (2n)! \geq 2^n(n!)^2 \quad \forall n \in \mathbb{N}$

9. Dimostrare che per ogni intero $n \geq 1$ valgono le seguenti disuguaglianze con i binomiali (pensare a quale relazione c'è tra queste disuguaglianze):

$$\binom{2n}{n} \leq 4^n, \qquad \binom{2n}{n} \leq \frac{4^n}{\sqrt{3n+1}}, \qquad \binom{2n}{n} \leq \frac{4^n}{\sqrt{3n}}.$$

$$\binom{2n}{n} \leq \frac{5^n}{\sqrt{3n+1}} \leq \frac{5^n}{\sqrt{3n}} \leq 5^n$$

PASSO BASE: $n=1$ $\quad \binom{2}{1} = \frac{2!}{1!1!} = 2 \leq \frac{5}{\sqrt{3+1}} = 2$

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PASSO INDUTTIVO: $\binom{2n}{n} \le \frac{4^n}{\sqrt{3n+1}} \Rightarrow \binom{2n+2}{n+1} \le \frac{4^{n+1}}{\sqrt{3n+4}}$

DIM: $\binom{2n+2}{n+1} = \frac{(2n+2)!}{[(n+1)!]^2} = \frac{\overset{2}{\cancel{(2n+2)}}(2n+1)}{(n+1)^{\cancel{2}}} \frac{(2n)!}{(n!)^2} \le$

$$\le \frac{2(2n+1)}{n+1} \frac{4^n}{\sqrt{3n+1}} \overset{?}{\le} \frac{4^{n+1}}{\sqrt{3n+4}}$$

$$\frac{\cancel{2}(2n+1)}{n+1} \frac{\overset{1}{\cancel{4^n}}}{\sqrt{3n+1}} \le \frac{\overset{2\ \cancel{4}}{\cancel{4^{n+1}}}}{\sqrt{3n+4}} \qquad \frac{(2n+1)}{n+1} \le \frac{2\sqrt{3n+1}}{\sqrt{3n+4}}$$

$$\frac{4n^2+4n+1}{n^2+2n+1} \le \frac{4(3n+1)}{3n+4}$$

$$\cancel{12n^3} + \cancel{12n^2} + 3n + \cancel{16n^2} + 16n + \cancel{4} \le \cancel{12n^3} + \cancel{24n^2} + 12n + \cancel{4n^2} + 8n + \cancel{4}$$

$$19n \le 20n \qquad \forall n \in \mathbb{N}$$

$$\rightsquigarrow \quad \binom{2n}{n} \le \frac{4^n}{\sqrt{3n+1}} \le \frac{4^n}{\sqrt{3n}} \le 4^n \qquad \forall n \ge 1$$

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