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Limiti 3

Argomenti: limiti di successioni **Difficoltà:** ⋆⋆⋆⋆

Prerequisiti: teoremi algebrici, teoremi di confronto, teoremi rapporto/radice

1. Calcolare, al variare del parametro reale α, i limiti delle seguenti successioni:
$$\frac{n^3+5n^\alpha+3}{n^4+7n+1}, \qquad \frac{n^2+3\sqrt{n}}{n+\sqrt{n^\alpha}}, \qquad \frac{n^2+2n+3}{n+5}-\alpha n.$$

2. Calcolare i limiti delle seguenti successioni:
$$\frac{|3n^2-n^3|-7n}{|6n-95|+5n^3}, \qquad \frac{||2-3n^4|-|16-12n^2|-8|}{(n+3)^5-n^5}, \qquad \frac{|n^3+(-1)^n n^2|}{|n^2+(-1)^n n^3|}.$$

3. (a) Calcolare i limiti delle seguenti successioni:
$$\sum_{k=n}^{2n}\frac{1}{k^2}, \qquad \sum_{k=n}^{2n}\frac{1}{\sqrt{k}}, \qquad \sum_{k=n^2}^{2n^2}\frac{1}{\sqrt[5]{k}}, \qquad \sum_{k=n}^{2n}\frac{1}{n+\sqrt{k}}.$$
   (b) Determinare per quali valori del parametro reale α le seguenti successioni tendono a zero:
$$n^\alpha\sum_{k=n}^{3n}\frac{1}{k^2}, \qquad \frac{1}{n^\alpha}\sum_{k=n}^{2n}\frac{1}{\sqrt[3]{k}}, \qquad n\sum_{k=n}^{3n}\frac{1}{k^\alpha}, \qquad n^\alpha\sum_{k=n}^{n^2}\frac{k^2}{2^k}.$$

4. Dimostrare che il seguente limite esiste ed è reale (più avanti nel corso si chiederà di calcolarlo esplicitamente):
$$\lim_{n\to+\infty}\sum_{k=n}^{2n}\frac{1}{k}.$$

5. Calcolare i limiti delle seguenti successioni, giustificando dettagliatamente i passaggi:
$$\sqrt[n]{4^n+n^4}, \qquad \sqrt[n]{4^n+3^n}, \qquad \sqrt[n]{4^n-3^n}, \qquad \sqrt[n]{n!-4^n}.$$

6. Calcolare, al variare del parametro reale α, i limiti delle seguenti successioni:
$$\sqrt[n]{\alpha^n+n^4}, \qquad \sqrt[n]{\alpha^n+4^n}, \qquad \sqrt[n]{\alpha^n+n!}, \qquad \sqrt[n]{\frac{4^n+3^n}{2^n+\alpha^n}}.$$

7. Dimostrare che le seguenti successioni sono ben definite e calcolarne il limite:
$$(\arctan n-\sin n)^{1/n}, \qquad \left(\arccos\frac{2014}{\sqrt{n}}-\sin n!\right)^{1/n}.$$

8. Calcolare, al variare del parametro reale positivo α, i limiti delle seguenti successioni:
$$\frac{(n!)^{1/\alpha}}{\alpha^n}, \qquad \frac{(n!)^\alpha}{\alpha^{n!}}, \qquad (n!)^{1/n^\alpha}, \qquad [(2n)!]^{1/n^\alpha}.$$

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1. Calcolare, al variare del parametro reale α, i limiti delle seguenti successioni:

$$\frac{n^3+5n^\alpha+3}{n^4+7n+1}, \qquad \frac{n^2+3\sqrt{n}}{n+\sqrt{n^\alpha}}, \qquad \frac{n^2+2n+3}{n+5}-\alpha n.$$

1.a) $$\frac{n^3+5n^\alpha+3}{n^5+7n+1} \to \begin{cases} 0 & \alpha<5 \\ 5 & \alpha=5 \\ +\infty & \alpha>5 \end{cases}$$

$$\alpha=5 \quad \frac{n^3+5n^5+3}{n^5+7n+1} = \frac{n^5}{n^5}\,\frac{\frac{1}{n}+5+\frac{3}{n^5}}{1+\frac{7}{n^3}+\frac{1}{n^5}} \to 5$$

$$\alpha<5 \quad \frac{n^3+5n^\alpha+3}{n^5+7n+1} = \frac{n^5}{n^5}\,\frac{\frac{1}{n}+\frac{5}{n^{(5-\alpha)}}+\frac{3}{n^5}}{1+\frac{7}{n^3}+\frac{1}{n^5}} \to 0$$

$$\alpha>5 \quad \frac{n^3+5n^\alpha+3}{n^5+7n+1} = \frac{n^5}{n^5}\,\frac{\frac{1}{n}+5n^{(\alpha-5)}+\frac{3}{n^5}}{1+\frac{7}{n^3}+\frac{1}{n^5}} \to +\infty$$

1.b) $$\frac{n^2+3\sqrt{n}}{n+\sqrt{n^\alpha}} \to \begin{cases} +\infty & \alpha<5 \\ 1 & \alpha=5 \\ 0 & \alpha>5 \end{cases}$$

1.c) $$\frac{n^2+2n+3}{n+5}-\alpha n \to \begin{cases} +\infty & \alpha<1 \\ -3 & \alpha=1 \\ -\infty & \alpha>1 \end{cases}$$

$$\frac{n^2+2n+3}{n+5}-\alpha n = \frac{n^2+2n+3-\alpha n^2-5\alpha n}{n+5} =$$

$$= \frac{(1-\alpha)n^2+(2-5\alpha)n+3}{n+5}$$

2. Calcolare i limiti delle seguenti successioni:

$$\frac{|3n^2 - n^3| - 7n}{|6n - 95| + 5n^3}, \qquad \frac{||2 - 3n^4| - |16 - 12n^2| - 8|}{(n+3)^5 - n^5}, \qquad \frac{|n^3 + (-1)^n n^2|}{|n^2 + (-1)^n n^3|}.$$

2.a) $$\frac{|3n^2 - n^3| - 7n}{|6n - 95| + 5n^3} \to 1/5$$

$$\frac{|3n^2 - n^3| - 7n}{|6n - 95| + 5n^3} = \frac{n^3}{n^3} \frac{\overbrace{\left|\frac{3}{n} - 1\right|}^{\leadsto 1} - \overbrace{\frac{7}{n^2}}^{\leadsto 0}}{\underbrace{\left|\frac{6}{n^2} - \frac{95}{n^3}\right|}_{\leadsto 0} + 5} \to 1/5$$

2.b) $$\frac{||2 - 3n^4| - |16 - 12n^2| - 8|}{(n+3)^5 - n^5} \to 1/5$$

$$\frac{||2 - 3n^4| - |16 - 12n^2| - 8|}{(n+3)^5 - n^5} = \frac{||2 - 3n^4| - |16 - 12n^2| - 8|}{15n^4 + 90n^3 + 270n^2 + 405n + 243} \to \frac{3}{15}$$

2.c) $$\frac{|n^3 + (-1)^n n^2|}{|n^2 + (-1)^n n^3|} \to 1$$

$$\frac{|n^3 + (-1)^n n^2|}{|n^2 + (-1)^n n^3|} = \frac{n^3}{n^3} \frac{\overbrace{\left|1 + \frac{(-1)^n}{n}\right|}^{\leadsto 1}}{\underbrace{\left|\frac{1}{n} + (-1)^n\right|}_{\leadsto 1}}$$

3. (a) Calcolare i limiti delle seguenti successioni:

$$\sum_{k=n}^{2n} \frac{1}{k^2}, \qquad \sum_{k=n}^{2n} \frac{1}{\sqrt{k}}, \qquad \sum_{k=n^2}^{2n^2} \frac{1}{\sqrt[5]{k}}, \qquad \sum_{k=n}^{2n} \frac{1}{n + \sqrt{k}}.$$

3.a) $$\sum_{k=n}^{2n} \frac{1}{k^2} \to 0$$

$$\sum_{k=n}^{2n} \frac{1}{k^2} = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2} \leq (n+1) \frac{1}{n^2} \to 0$$

3.b) $\sum_{k=n}^{2n} \frac{1}{\sqrt{k}} \to +\infty$

$$\sum_{k=n}^{2n} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \dots + \frac{1}{\sqrt{2n}} \geq (n+1) \frac{1}{\sqrt{2n}} \to +\infty$$

3.c) $\sum_{k=n^2}^{2n^2} \frac{1}{\sqrt[5]{k}} \to +\infty$

$$\sum_{k=n^2}^{2n^2} \frac{1}{\sqrt[5]{k}} = \frac{1}{\sqrt[5]{n^2}} + \frac{1}{\sqrt[5]{(n+1)^2}} + \dots + \frac{1}{\sqrt[5]{2n^2}} \geq (n^2+1) \frac{1}{\sqrt[5]{2n^2}} \to +\infty$$

3.d) $\sum_{k=n}^{2n} \frac{1}{n+\sqrt{k}} \to 1$

$$\sum_{k=n}^{2n} \frac{1}{n+\sqrt{k}} = \frac{1}{n+\sqrt{n}} + \frac{1}{n+\sqrt{n+1}} + \dots + \frac{1}{n+\sqrt{2n}} \to 1$$

$$\underbrace{(n+1) \frac{1}{n+\sqrt{2n}}}_{\to 1} \leq \underbrace{\sum_{k=n}^{2n} \frac{1}{n+\sqrt{k}}}_{\to 1} \leq \underbrace{(n+1) \frac{1}{n+\sqrt{n}}}_{\to 1}$$

(b) Determinare per quali valori del parametro reale α le seguenti successioni tendono a zero:

$$n^\alpha \sum_{k=n}^{3n} \frac{1}{k^2}, \qquad \frac{1}{n^\alpha} \sum_{k=n}^{2n} \frac{1}{\sqrt[3]{k}}, \qquad n \sum_{k=n}^{3n} \frac{1}{k^\alpha}, \qquad n^\alpha \sum_{k=n}^{n^2} \frac{k^2}{2^k}.$$

3.e) $n^\alpha \sum_{k=n}^{3n} \frac{1}{k^2} \to 0 \quad \alpha < 1$

$$n^\alpha \sum_{k=n}^{3n} \frac{1}{k^2} = n^\alpha \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(3n)^2} \right) \leq n^\alpha (2n+1) \frac{1}{n^2} =$$

$$= \frac{2n^{\alpha+1} + n^\alpha}{n^2} \to 0 \quad \alpha + 1 < 2 \quad \rightsquigarrow \quad \alpha < 1$$

$\alpha = 1$

$$\underbrace{n (2n+1) \frac{1}{(3n)^2}}_{\to 2/9} \leq n \sum_{k=n}^{3n} \frac{1}{k^2} \leq \underbrace{n (2n+1) \frac{1}{n^2}}_{\to 2}$$

$\alpha > 1 \qquad m^{\alpha} \sum_{k=m}^{3m} \frac{1}{k^2} \geq m^{\alpha}(2m+1)\frac{1}{(3m)^2} \to +\infty$

3.f) $\frac{1}{m^{\alpha}} \sum_{k=m}^{2m} \frac{1}{\sqrt[3]{k}} \to 0 \qquad \alpha > 2/3$

$$\frac{1}{m^{\alpha}} \sum_{k=m}^{2m} \frac{1}{\sqrt[3]{k}} = \frac{1}{m^{\alpha}}\left(\frac{1}{\sqrt[3]{m}} + \frac{1}{\sqrt[3]{m+1}} + \dots + \frac{1}{\sqrt[3]{2m}}\right) \leq \frac{1}{m^{\alpha}}(m+1)\frac{1}{\sqrt[3]{m}} =$$

$$= \frac{m+1}{m^{\alpha+1/3}} \to 0 \qquad \alpha + 1/3 > 1 \qquad \alpha > 2/3$$

$\alpha = 2/3 \qquad \underbrace{\frac{1}{m^{2/3}}(m+1)\frac{1}{\sqrt[3]{2m}}}_{\to 1/\sqrt[3]{2}} \leq \frac{1}{m^{2/3}} \sum_{k=m}^{2m} \frac{1}{\sqrt[3]{k}} \leq \underbrace{\frac{1}{m^{2/3}}(m+1)\frac{1}{\sqrt[3]{m}}}_{\to 1}$

$\alpha < 2/3 \qquad \frac{1}{m^{\alpha}} \sum_{k=m}^{2m} \frac{1}{\sqrt[3]{k}} \geq \frac{1}{m^{\alpha}}(m+1)\frac{1}{\sqrt[3]{m}} \to +\infty$

3.g) $m \sum_{k=m}^{3m} \frac{1}{k^{\alpha}} \to 0 \qquad \alpha > 2$

$$m \sum_{k=m}^{3m} \frac{1}{k^{\alpha}} = m\left(\frac{1}{m^{\alpha}} + \frac{1}{(m+1)^{\alpha}} + \dots + \frac{1}{(3m)^{\alpha}}\right) \leq m(2m+1)\frac{1}{m^{\alpha}} =$$

$$= \frac{2m^2+m}{m^{\alpha}} \to 0 \qquad \alpha > 2$$

$\alpha = 2 \qquad \underbrace{m(2m+1)\frac{1}{(3m)^2}}_{\to 2/9} \leq m \sum_{k=m}^{3m} \frac{1}{k^{\alpha}} \leq \underbrace{m(2m+1)\frac{1}{m^2}}_{\to 2}$

$\alpha < 2 \qquad m \sum_{k=m}^{3m} \frac{1}{k^{\alpha}} \geq m(2m+1)\frac{1}{(3m)^{\alpha}} \to +\infty$

3.b) $n^\alpha \sum_{k=n}^{n^2} \frac{k^2}{2^k} \to 0 \quad \forall \alpha \in \mathbb{R}$

$$n^\alpha \sum_{k=n}^{n^2} \frac{k^2}{2^k} = n^\alpha \left(\frac{n^2}{2^n} + \frac{(n+1)^2}{2^{n+1}} + \dots + \frac{(n^2)^2}{2^{n^2}} \right) \underset{n\geq 3}{\leq} n^\alpha (n^2-n+1) \frac{n^2}{2^n} =$$

$$= \frac{n^{4+\alpha} - n^{3+\alpha} + n^{2+\alpha}}{2^n} \to 0 \quad \forall \alpha \in \mathbb{R}$$

4. Dimostrare che il seguente limite esiste ed è reale (più avanti nel corso si chiederà di calcolarlo esplicitamente):

$$\lim_{n\to+\infty} \sum_{k=n}^{2n} \frac{1}{k}.$$

$$\lim_{n\to+\infty} \sum_{k=n}^{2n} \frac{1}{k} \to \lambda \qquad \frac{1}{2} \leq \lambda \leq 1$$

$$\underbrace{\frac{n+1}{2n}}_{\to 1/2} \leq \sum_{k=n}^{2n} \frac{1}{k} = \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n} \leq \underbrace{\frac{n+1}{n}}_{\to 1}$$

5. Calcolare i limiti delle seguenti successioni, giustificando dettagliatamente i passaggi:

$$\sqrt[n]{4^n + n^4}, \qquad \sqrt[n]{4^n + 3^n}, \qquad \sqrt[n]{4^n - 3^n}, \qquad \sqrt[n]{n! - 4^n}.$$

5.a) $\sqrt[n]{4^n + n^4} \to 4$

$$\sqrt[n]{4^n + n^4} = \sqrt[n]{b_n} \qquad \frac{b_{n+1}}{b_n} = \frac{4^{n+1} + (n+1)^4}{4^n + n^4} \to 4$$

5.b) $\sqrt[n]{4^n + 3^n} \to 4$

$$\sqrt[n]{4^n + 3^n} = \sqrt[n]{b_n} \qquad \frac{b_{n+1}}{b_n} = \frac{4^{n+1} + 3^{n+1}}{4^n + 3^n} \to 4$$

5.c) $\sqrt[n]{4^n - 3^n} \to 4$

$$\sqrt[n]{4^n - 3^n} = \sqrt[n]{b_n} \qquad \frac{b_{n+1}}{b_n} = \frac{4^{n+1} - 3^{n+1}}{4^n - 3^n} \to 4$$

5.d) $\sqrt[n]{n! - 5^n} \to +\infty$

$$\sqrt[n]{n! - 5^n} = \sqrt[n]{b_n} \qquad \frac{b_{n+1}}{b_n} = \frac{(n+1)! - 5^{n+1}}{n! - 5^n} =$$

$$= \frac{n!}{n!} \frac{(n+1) - \frac{5^{n+1}}{n!}}{1 - \frac{5^{n+1}}{n!}} \to +\infty$$

6. Calcolare, al variare del parametro reale α, i limiti delle seguenti successioni:

$$\sqrt[n]{\alpha^n + n^4}, \qquad \sqrt[n]{\alpha^n + 4^n}, \qquad \sqrt[n]{\alpha^n + n!}, \qquad \sqrt[n]{\frac{4^n + 3^n}{2^n + \alpha^n}}.$$

6.a) $\sqrt[n]{\alpha^n + n^5} \to \begin{cases} 1 & |\alpha| \leq 1 \\ \alpha & \alpha > 1 \\ \text{N.E.} & \alpha < -1 \end{cases}$

$$|\alpha| \leq 1 \quad \sqrt[n]{\alpha^n + n^5} = \underbrace{\sqrt[n]{n^5}}_{\to 1} \; \underbrace{\sqrt[n]{\underbrace{\frac{\alpha^n}{n^5}}_{\to 0} + 1}}_{\to 1} \to 1$$

$$\alpha > 1 \quad \sqrt[n]{\alpha^n + n^5} = \alpha \underbrace{\sqrt[n]{1 + \frac{n^5}{\alpha^n}}}_{\to 1} \to \alpha$$

$$\alpha < -1 \quad \sqrt[n]{\alpha^n + n^5} = \sqrt[n]{\alpha^n} \underbrace{\sqrt[n]{1 + \frac{n^5}{\alpha^n}}}_{\to 1} \to \begin{cases} |\alpha| & n \text{ PARI} \\ -|\alpha| & n \text{ DISPARI} \end{cases}$$

6.b) $\sqrt[n]{\alpha^n + 5^n} \to \begin{cases} 5 & |\alpha| < 5 \\ \alpha & \alpha \geq 5 \\ \text{N.E.} & \alpha \leq -5 \end{cases}$

$$|\alpha| < 5 \quad \sqrt[n]{\alpha^n + 5^n} = 5 \underbrace{\sqrt[n]{\frac{\alpha^n}{5^n} + 1}}_{\to 1} \to 5$$

$$\alpha \geq 5 \quad \sqrt[n]{\alpha^n + 5^n} = \alpha \underbrace{\sqrt[n]{1 + \frac{5^n}{\alpha^n}}}_{\to 1} \to \alpha$$

$$\alpha < -5 \qquad \sqrt[n]{\alpha^n + 5^n} = \sqrt[n]{\alpha^n}\,\overbrace{\sqrt[n]{1 + \frac{5^n}{\alpha^n}}}^{\to 1} \to \begin{cases} |\alpha| & n \text{ PARI} \\ -|\alpha| & n \text{ DISPARI} \end{cases}$$

$$\alpha = -5 \qquad \sqrt[n]{\alpha^n + 5^n} = \begin{cases} 5\sqrt[n]{2} & n \text{ PARI} & \to 5 \\ 0 & n \text{ DISPARI} & \to 0 \end{cases}$$

6.c) $\sqrt[n]{\alpha^n + n!} \to +\infty \quad \forall \alpha \in \mathbb{R}$

$$\sqrt[n]{\alpha^n + n!} = \overbrace{\sqrt[n]{n!}}^{\to +\infty}\,\overbrace{\sqrt[n]{\frac{\alpha^n}{n!} + 1}}^{\to 1} \to +\infty \quad \forall \alpha \in \mathbb{R}$$

6.d) $\sqrt[n]{\dfrac{5^n + 3^n}{2^n + \alpha^n}} \to \begin{cases} 2 & |\alpha| < 2 \\ 5/\alpha & \alpha \geq 2 \\ N.E. & \alpha \leq -2 \end{cases}$

$$|\alpha| < 2 \qquad \sqrt[n]{\frac{5^n + 3^n}{2^n + \alpha^n}} = \frac{5}{2}\,\overbrace{\sqrt[n]{\frac{1 + \frac{3^n}{5^n}}{1 + \frac{\alpha^n}{2^n}}}}^{\to 1} \to 2$$

$$\alpha \geq 2 \qquad \sqrt[n]{\frac{5^n + 3^n}{2^n + \alpha^n}} = \frac{5}{\alpha}\,\overbrace{\sqrt[n]{\frac{1 + \frac{3^n}{5^n}}{\frac{2^n}{\alpha^n} + 1}}}^{\to 1} \to 5/\alpha$$

$$\alpha < -2 \qquad \sqrt[n]{\frac{5^n + 3^n}{2^n + \alpha^n}} = \frac{5}{\sqrt[n]{\alpha^n}}\,\overbrace{\sqrt[n]{\frac{1 + \frac{3^n}{5^n}}{\frac{2^n}{\alpha^n} + 1}}}^{\to 1} \to \begin{cases} 5/|\alpha| & n \text{ PARI} \\ -5/|\alpha| & n \text{ DISPARI} \end{cases}$$

$$\alpha = -2 \qquad \sqrt[n]{\frac{5^n + 3^n}{2^n + \alpha^n}} = \begin{cases} \sqrt[n]{\dfrac{5^n + 3^n}{2 \cdot 2^n}} & n \text{ PARI} \\ N.D. & n \text{ DISPARI} \end{cases}$$

7. Dimostrare che le seguenti successioni sono ben definite e calcolarne il limite:

$$(\arctan n - \sin n)^{1/n}, \qquad \left(\arccos \frac{2014}{\sqrt{n}} - \sin n!\right)^{1/n}.$$

7.a) $(\mathrm{ARCTAN}\, n - \mathrm{SIN}\, n)^{1/n} \to 1$

DEF. $\mathrm{ARCTAN}\, n \geq 1{,}1 \to \mathrm{ARCTAN}\, n - \mathrm{SIN}\, n \geq 0{,}1$

$$\underbrace{(0{,}1)^{1/n}}_{\to 1} \leq \underbrace{(\mathrm{ARCTAN}\, n - \mathrm{SIN}\, n)^{1/n}}_{\to 1} \leq \underbrace{(2{,}15)^{1/n}}_{\to 1}$$

7.b) $\left(\mathrm{ARCCOS}\, \frac{2015}{\sqrt{n}} - \mathrm{SIN}\, n!\right)^{1/n} \to 1$

DEF. $\mathrm{ARCCOS}\, \frac{2015}{\sqrt{n}} \geq 1{,}1 \to \mathrm{ARCCOS}\, \frac{2015}{\sqrt{n}} - \mathrm{SIN}\, n! \geq 0{,}1$

$$\underbrace{(0{,}1)^{1/n}}_{\to 1} \leq \underbrace{\left(\mathrm{ARCCOS}\, \frac{2015}{\sqrt{n}} - \mathrm{SIN}\, n!\right)^{1/n}}_{\to 1} \leq \underbrace{(2{,}15)^{1/n}}_{\to 1}$$

8. Calcolare, al variare del parametro reale positivo α, i limiti delle seguenti successioni:

$$\frac{(n!)^{1/\alpha}}{\alpha^n}, \qquad \frac{(n!)^{\alpha}}{\alpha^{n!}}, \qquad (n!)^{1/n^\alpha}, \qquad [(2n)!]^{1/n^\alpha}.$$

8.a) $\frac{(n!)^{1/\alpha}}{\alpha^n}$

$$\sqrt[n]{a_n} = \frac{\left(\sqrt[n]{n!}\right)^{1/\alpha}}{\alpha} \to +\infty$$

8.b) $\frac{(n!)^{\alpha}}{\alpha^{n!}} \to \begin{cases} +\infty & \alpha \leq 1 \\ 0 & \alpha > 1 \end{cases}$

$$\frac{a_{n+1}}{a_n} = \left(\frac{(n+1)!}{n!}\right)^{\alpha} \frac{\alpha^{n!}}{\alpha^{(n+1)!}} = (n+1)^{\alpha} \cdot \alpha^{n! - (n+1)!} =$$

$$= (n+1)^{\alpha}\, \alpha^{n!(1-n-1)} = (n+1)^{\alpha}\, \alpha^{-n \cdot n!}$$

$$\alpha > 1 \qquad (n+1)^2 \, \alpha^{-n \cdot n!} = \frac{(n+1)^2}{\alpha^{n \cdot n!}} \to 0$$

$$\alpha \leq 1 \qquad (n+1)^2 \, \beta^{n \cdot n!} \to +\infty \qquad (\beta = 1/\alpha \geq 1)$$

8.c) $(n!)^{1/n^\alpha} \to \begin{cases} 1 & \alpha > 1 \\ +\infty & \alpha \leq 1 \end{cases}$

$$(n!)^{1/n^\alpha} = e^{\frac{\log n!}{n^\alpha}} \to \begin{cases} 1 & \alpha > 1 \\ +\infty & \alpha \leq 1 \end{cases}$$

$$\frac{\log n!}{n^\alpha} \leq \frac{n \log n}{n^\alpha} = \frac{\log n}{n^{\alpha-1}} \to 0 \qquad \alpha > 1$$

$$\alpha \leq 1 \qquad \frac{\log n!}{n^\alpha} = \frac{\log \sqrt[n]{n!}}{n^{\alpha-1}} = n^{1-\alpha} \log \sqrt[n]{n!} \to +\infty$$

8.d) $[(2n)!]^{1/n^\alpha} \to \begin{cases} 1 & \alpha < 1 \\ +\infty & \alpha \geq 1 \end{cases}$

$$[(2n)!]^{1/n^\alpha} = e^{\frac{\log (2n)!}{n^\alpha}} \to \begin{cases} 1 & \alpha > 1 \\ +\infty & \alpha \leq 1 \end{cases}$$

$$\frac{\log (2n)!}{n^\alpha} \leq \frac{2n \log 2n}{n^\alpha} = \frac{2 \log 2n}{n^{\alpha-1}} \to 0 \qquad \alpha > 1$$

$$\alpha \leq 1 \qquad \frac{\log (2n)!}{n^\alpha} = \frac{\log \sqrt[n]{(2n)!}}{n^{\alpha-1}} = n^{1-\alpha} \log \sqrt[n]{(2n)!} \to +\infty$$