Prodotti scalari 2

Argomenti: prodotti scalari generali **Difficoltà:** ⋆⋆⋆

Prerequisiti: prodotti scalari, forme quadratiche, Gram-Schmidt

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Nei seguenti punti vengono presentate delle espressioni che coinvolgono due elementi generici di uno spazio vettoriale. Per ciascuna di essa si richiede di

(a) stabilire se definisce un prodotto scalare oppure no, ed in caso negativo specificare quali proprietà vengono a mancare,

(b) nel caso in cui si tratta di un prodotto scalare, determinare la matrice che lo rappresenta rispetto alla base canonica dello spazio in questione e stabilirne la segnatura,

(c) nel caso in cui si tratta di un prodotto scalare definito positivo, determinare una base ortonormale (ovviamente rispetto a quel prodotto) dello spazio vettoriale di partenza.

1. Spazio vettoriale: $\mathbb{R}^3$. Elementi generici: (x_1, y_1, z_1) e (x_2, y_2, z_2). Espressioni:

$$x_1x_2 + y_1y_2 + z_1y_2 + z_2y_1, \qquad x_1x_2 + y_1y_2 + z_1y_2 + z_2y_1 + 5z_1z_2,$$

$$x_1y_1, \qquad x_1x_2, \qquad x_1y_2, \qquad x_1y_2 + x_2y_1, \qquad x_1x_2 + y_1y_2, \qquad x_1y_1 + x_2y_2,$$

$$(x_1 + y_1)(x_2 + y_2), \qquad (x_1 + y_1)(x_2 + y_2) + 10z_1z_2, \qquad (x_1 + y_1 + z_1)(x_2 + y_2 + z_2),$$

$$(x_1 + x_2)(y_1 + y_2), \qquad (2x_1 + y_1)(x_2 + 2y_2), \qquad (x_1 + y_1 + z_1)(x_2 + y_2).$$

2. Spazio vettoriale: $\mathbb{R}_{\leq 2}[x]$ con base canonica $\{1, x, x^2\}$. Elementi generici: $p(x)$ e $q(x)$. Espressioni:

$$\int_{-1}^{1} p(x)q(x)\,dx, \qquad \int_{0}^{1} p(x)q(x)\,dx, \qquad \int_{-1}^{1} p'(x)q'(x)\,dx, \qquad \int_{0}^{1} p(x)q'(x)\,dx,$$

$$p(x)q(x), \qquad p(0)q(0), \qquad p(1)q(0), \qquad p(0)q(1) + p(1)q(0), \qquad p'(0)q'(0),$$

$$p(0)q(0) + p(1)q(1) + p(-1)q(-1), \qquad p(0)q(0) + p'(1)q'(1) + p(-1)q(-1),$$

$$(p(0) + 2p(1))(q(0) + 2q(1)), \qquad (p(0) + 2q(1))(q(0) + 2p(1)), \qquad p(0)q(0) - p(1)q(1),$$

$$\int_{-1}^{1} p'(x)q'(x)\,dx + p(3)q(3), \qquad \int_{-1}^{1} p(x)q(x)\,dx - p(2)q(2).$$

3. Spazio vettoriale: $\mathbb{R}_{\leq 3}[x]$ con base canonica $\{1, x, x^2, x^3\}$. Elementi generici: $p(x)$ e $q(x)$. Stesse espressioni del punto precedente.

4. Spazio vettoriale: $M_{2\times 2}$. Elementi generici: A e B. Espressioni:

$$AB, \qquad AB + BA, \qquad \mathrm{Tr}(AB), \qquad \mathrm{Tr}(A)\cdot\mathrm{Tr}(B), \qquad \mathrm{Tr}(A^tB), \qquad \mathrm{Tr}(AB^t),$$

$$\det(AB), \qquad (1,0)AB\begin{pmatrix}0\\1\end{pmatrix}, \qquad (1,0)(AB+BA)\begin{pmatrix}0\\1\end{pmatrix}, \qquad \det A \cdot \det B.$$

SOLUZIONI ?
BY GIMUSI 02/02/2015

1. Spazio vettoriale: $\mathbb{R}^3$. Elementi generici: (x_1, y_1, z_1) e (x_2, y_2, z_2). Espressioni:

1) $x_1x_2 + y_1y_2 + z_1y_2 + z_2y_1$, 2) $x_1x_2 + y_1y_2 + z_1y_2 + z_2y_1 + 5z_1z_2$,

3) x_1y_1, 4) x_1x_2, 5) x_1y_2, 6) $x_1y_2 + x_2y_1$, 7) $x_1x_2 + y_1y_2$, 8) $x_1y_1 + x_2y_2$,

9) $(x_1 + y_1)(x_2 + y_2)$, 10) $(x_1 + y_1)(x_2 + y_2) + 10z_1z_2$, 11) $(x_1 + y_1 + z_1)(x_2 + y_2 + z_2)$,

12) $(x_1 + x_2)(y_1 + y_2)$, 13) $(2x_1 + y_1)(x_2 + 2y_2)$, 15) $(x_1 + y_1 + z_1)(x_2 + y_2)$.

1) $x_1x_2+y_1y_2+z_1y_2+z_2y_1$

(a) (i) SIMMETRIA: $x_1x_2+y_1y_2+z_1y_2+z_2y_1 = x_2x_1+y_2y_1+z_2y_1+z_1y_2$ SÍ

(ii) LINEARITÁ: $(x_1+\hat{x}_1)x_2+(y_1+\hat{y}_1)y_2+(z_1+\hat{z}_1)z_2+z_2(y_1+\hat{y}_1)=$
$=(x_1x_2+y_1y_2+z_1y_2+z_2y_1)+(\hat{x}_1x_2+\hat{y}_1y_2+\hat{z}_1y_2+z_2\hat{y}_1)$ SÍ

(iii) " : $\lambda x_1x_2+\lambda y_1y_2+\lambda z_1y_2+\lambda z_2y_1=\lambda(x_1x_2+y_1y_2+z_1y_2+z_2y_1)$ SÍ

(b)

$$B=\begin{pmatrix}1&0&0\\0&1&1\\0&1&0\end{pmatrix} \qquad DET(B)= 1\cdot(-1) = -1$$

SYLVESTER (1,2,3): $DET(1)=1$ $DET\begin{pmatrix}1&0\\0&1\end{pmatrix}=1$ $DET(B)=-1$

$\overset{P}{+}\ \overset{P}{+}\ \overset{V}{+}\ -$ $n_+=2$ $n_-=1$ $n_0=0$ $\rightsquigarrow$ INDEFINITA

2) $x_1x_2+y_1y_2+z_1y_2+z_2y_1+5z_1z_2$

(a) (i) SIMM. SÍ (ii) LINEARITÁ SÍ (iii) LINEARITÁ SÍ

(b)

$$B=\begin{pmatrix}1&0&0\\0&1&1\\0&1&5\end{pmatrix} \qquad DET(B)= 1\cdot(5) = 5$$

SYLVESTER (1,2,3): $DET(1)=1$ $DET\begin{pmatrix}1&0\\0&1\end{pmatrix}=1$ $DET(B)=5$

$\overset{P}{+}\ \overset{P}{+}\ \overset{P}{+}\ +$ $n_+=3$ $n_-=0$ $n_0=0$ $\rightsquigarrow$ DEFINITA POSITIVA

(c) $V_1=(1,0,0)$ $V_2=(0,1,0)$ $V_3=(0,0,1)$

$W_1=V_1$ $\langle W_1,W_1\rangle=(1\,0\,0)\begin{pmatrix}1\\0\\0\end{pmatrix}=1$ $\langle W_1,W_2\rangle=(1\,0\,0)\begin{pmatrix}0\\1\\0\end{pmatrix}=0$

$W_1 = V_1 \quad \langle W_1, W_1\rangle = (0\ 1\ 1)\begin{pmatrix}0\\1\\0\end{pmatrix} = 1 \quad \langle W_1, V_3\rangle = (1\ 0\ 0)\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$

$$\langle W_2, V_3\rangle = (0\ 1\ 1)\begin{pmatrix}0\\0\\1\end{pmatrix} = 1$$

$$W_3 = V_3 - \frac{\langle W_1, V_3\rangle}{\langle W_1, W_1\rangle} W_1 - \frac{\langle W_2, V_3\rangle}{\langle W_2, W_2\rangle} W_2 = V_3 - W_2 = (0,-1,1)$$

$$\langle W_3, W_3\rangle = (0\ 0\ 5)\begin{pmatrix}0\\-1\\1\end{pmatrix} = 5$$

$w_1^* = \frac{W_1}{\|W_1\|} = (1,0,0) \quad w_2^* = \frac{W_2}{\|W_2\|} = (0,1,0) \quad w_3^* = \frac{W_3}{\|W_3\|} = (0,-\frac{1}{2},\frac{1}{2})$

3) $x_1 y_1$

(a) (i) SIMM. $x_1 y_1 \neq x_2 y_2$ NO

4) $x_1 x_2$

(a) (i) SIMM. $x_1 x_2 = x_2 x_1$ SÍ

(ii) LIN. $(x_1 + \hat{x}_1) x_2 = x_1 x_2 + \hat{x}_1 x_2$ SÍ

(iii) LIN. $(\lambda x_1) x_2 = \lambda (x_1 x_2)$ SÍ

(b)

$$B = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix} \qquad n_+ = 1 \quad n_- = 0 \quad n_0 = 2 \leadsto \text{SEMIDEF. POSITIVA}$$

5) $x_1 y_2$

(a) (i) SIMM. $x_1 y_2 \neq x_2 y_1$ NO

6) $x_1 y_2 + x_2 y_1$

(a) (i) SIMM. $x_1 y_2 + x_2 y_1 = x_2 y_1 + x_1 y_2$ SÍ

(ii) LIN. $(x_1 + \hat{x}_1) y_2 + x_2 (y_1 + \hat{y}_1) = (x_1 y_2 + x_2 y_1) + (\hat{x}_1 y_2 + x_2 \hat{y}_1)$ SÍ

(iii) LIN. $(\lambda x_1) y_2 + x_2 (\lambda y_1) = \lambda (x_1 y_2 + x_2 y_1)$ SÍ

(b)

$$B = \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{pmatrix} \quad |B - \lambda I| = \begin{vmatrix}-\lambda & 1 & 0\\1 & -\lambda & 0\\0 & 0 & -\lambda\end{vmatrix} = -\lambda(\lambda^2 - 1) = -\lambda^3 + \lambda$$

CARTESIO: $-\lambda^3 + \lambda \quad n_0 = 1 \quad n_+ = 1 \quad n_- = 3-1-1 = 1 \leadsto$ INDEFINITA

7) $x_1 x_2 + y_1 y_2$

(a) (i) SIMM. SÌ (ii) LINEARITÀ SÌ (iii) LINEARITÀ SÌ

(b)
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad n_+ = 2 \quad n_- = 0 \quad n_0 = 1 \quad \rightsquigarrow$$ SEMIDEF. POSITIVA

8) $x_1 y_1 + x_2 y_2$

(a) (i) SIMM. $x_2 y_2 + x_1 y_1 = x_1 y_1 + x_2 y_2$ SÌ

(ii) LIN. $(x_1 + \hat{x}_1)(y_1 + \hat{y}_1) + x_2 y_2 = x_1 y_1 + x_1 \hat{y}_1 + \hat{x}_1 y_1 + \hat{x}_1 \hat{y}_1 + x_2 y_2$ NO

(iii) LIN. $(\lambda x_1)(\lambda y_1) + x_2 y_2 = \lambda^2 x_1 y_1 + x_2 y_2$ NO

9) $(x_1 + y_1)(x_2 + y_2) = x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2$

(a) (i) SIMM. $(x_2 + y_2)(x_1 + y_1) = (x_1 + y_1)(x_2 + y_2)$ SÌ

(ii) LIN. $(x_1 + \hat{x}_1 + y_1 + \hat{y}_1)(x_2 + y_2) = (x_1 + y_1)(x_2 + y_2) + (\hat{x}_1 + \hat{y}_1)(x_2 + y_2)$ SÌ

(iii) LIN. $(\lambda x_1 + \lambda y_1)(x_2 + y_2) = \lambda (x_1 + y_1)(x_2 + y_2)$ SÌ

(b)
$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad |B - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda \left[(1-\lambda)^2 - 1 \right] = -\lambda^3 + 2\lambda^2$$

CARTESIO: $-\lambda^3 + 2\lambda^2$ $n_0 = 2$ $n_+ = 1$ $n_- = 3 - 2 - 1 = 0$ SEMIDEF. POSITIVA

10) $(x_1 + y_1)(x_2 + y_2) + 10 z_1 z_2 = x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2 + 10 z_1 z_2$

(a) (i) SIMM. SÌ (ii) LINEARITÀ SÌ (iii) LINEARITÀ SÌ

(b)
$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix} \quad |B - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 10-\lambda \end{vmatrix} = \begin{cases} (10-\lambda)\left[(1-\lambda)^2 - 1\right] = \\ (10-\lambda)(\lambda^2 - 2\lambda) = \\ = -\lambda^3 + 12\lambda^2 - 20\lambda \end{cases}$$

CART: $-\lambda^3 + 12\lambda^2 - 20\lambda$ $n_0 = 1$ $n_+ = 2$ $n_- = 3 - 2 - 1 = 0$ SEMIDEF. POSITIVA

11) $(x_1 + y_1 + z_1)(x_2 + y_2 + z_2) = x_1 x_2 + x_1 y_2 + x_1 z_2 + y_1 x_2 + y_1 y_2 + y_1 z_2 + z_1 x_2 + z_1 y_2 + z_1 z_2$

(a) (i) SIMM. SÌ (ii) LINEARITÀ SÌ (iii) LINEARITÀ SÌ

(b)
$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

COMPL. QUAD. $(x+y+z)^2$ $n_+ = 1$ $n_0 = 2$ $n_- = 0$

$\rightsquigarrow$ SEMIDEF. POSITIVA

12) $(x_1+x_2)(y_1+y_2) = x_1y_1+x_1y_2+x_2y_1+x_2y_2$

(a) (i) SIMM. $(x_2+x_1)(y_2+y_1) = (x_1+x_2)(y_1+y_2)$ SÍ

(ii) LIN. $(x_1+\hat{x}_1+x_2)(y_1+\hat{y}_1+y_2) = (x_1+x_2)(y_1+y_2)+\hat{x}_1(x_1+x_2)+\hat{y}_1(y_1+y_2)+\hat{x}_1\hat{y}_1$ NO

(iii) LIN. $(\lambda x_1+x_2)(\lambda y_1+y_2) = \lambda^2 x_1y_1+\lambda x_1y_2+\lambda x_2y_1+x_2y_2$ NO

13) $(2x_1+y_1)(x_2+2y_2) = 2x_1x_2+4x_1y_2+y_1x_2+2y_1y_2$

(a) (i) SIMM. $(2x_2+y_2)(x_1+2y_1) \neq (2x_1+y_1)(x_2+2y_2)$ NO

(ii) LIN. $(2x_1+2\hat{x}_1+2y_1+2\hat{y}_1)(x_2+2y_2)$ $(2x_1+y_1)(x_2+2y_2)+(2\hat{x}_1+\hat{y}_1)(x_2+2y_2)$ SÍ

(iii) LIN. $(2\lambda x_1+\lambda y_1)(x_2+2y_2) = \lambda(2x_1+y_1)(x_2+2y_2)$ SÍ

14) $(x_1+y_1+z_1)(x_2+y_2)$

(a) (i) SIMM. $(x_2+y_2+z_2)(x_1+y_1) \neq (x_1+y_1+z_1)(x_2+y_2)$ NO

(ii) LIN. SÍ (iii) LIN. SÍ

2. Spazio vettoriale: $\mathbb{R}_{\leq 2}[x]$ con base canonica $\{1, x, x^2\}$. Elementi generici: $p(x)$ e $q(x)$. Espressioni:

1) $\int_{-1}^{1} p(x)q(x)\,dx$, 2) $\int_{0}^{1} p(x)q(x)\,dx$, 3) $\int_{-1}^{1} p'(x)q'(x)\,dx$, 4) $\int_{0}^{1} p(x)q'(x)\,dx$,

5) $p(x)q(x)$, 6) $p(0)q(0)$, 7) $p(1)q(0)$, 8) $p(0)q(1)+p(1)q(0)$, 9) $p'(0)q'(0)$,

10) $p(0)q(0)+p(1)q(1)+p(-1)q(-1)$, 11) $p(0)q(0)+p'(1)q'(1)+p(-1)q(-1)$,

12) $(p(0)+2p(1))(q(0)+2q(1))$, 13) $(p(0)+2q(1))(q(0)+2p(1))$, 14) $p(0)q(0)-p(1)q(1)$,

15) $\int_{-1}^{1} p'(x)q'(x)\,dx + p(3)q(3)$, 16) $\int_{-1}^{1} p(x)q(x)\,dx - p(2)q(2)$.

1) $\int_{-1}^{1} p(x)q(x)\,dx$

(a) (i) SIMM. $\int pq = \int qp$ SÍ

(ii) LIN. $\int \lambda p\cdot q = \lambda \int p\cdot q$ SÍ

(iii) LIN. $\int (p_1+p_2)q = \int p_1q + \int p_2 q$ SÍ

(b)

$$B = \begin{pmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \\ \langle x,1\rangle & \langle x,x\rangle & \langle x,x^2\rangle \\ \langle x^2,1\rangle & \langle x^2,x\rangle & \langle x^2,x^2\rangle \end{pmatrix}$$

$$\begin{cases} \langle 1,1\rangle = \int_{-1}^{1} dx = [x]_{-1}^{1} = 2 & \langle 1,x\rangle = \int_{-1}^{1} x\,dx = [x^2/2]_{-1}^{1} = 0 \\ \langle 1,x^2\rangle = \int_{-1}^{1} x^2 dx = [x^3/3]_{-1}^{1} = 2/3 & \langle x,x\rangle = \int_{-1}^{1} x^2 dx = 2/3 \\ \langle x,x^2\rangle = \int_{-1}^{1} x^3 dx = [x^4/4]_{-1}^{1} = 0 & \langle x^2,x^2\rangle = \int_{-1}^{1} x^4 dx = [x^5/5]_{-1}^{1} = 2/5 \end{cases}$$

$$\rightsquigarrow B = \begin{pmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{pmatrix} \quad DET(B) = \frac{8}{15} - \frac{8}{27} = \frac{216-120}{405} = \frac{\overset{32}{\cancel{96}}}{\underset{135}{\cancel{405}}} = \frac{32}{135}$$

SYLVESTER $(1,2,3)$: $DET(2) = 2 \quad DET\begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix} = 4/3 \quad DET(B) = 32/135$

$$\overset{P}{+}\overset{P}{+}\overset{P}{+}+ \quad m_+ = 3 \quad m_- = 0 \quad m_0 = 0 \rightsquigarrow \text{DEF. POSITIVA}$$

(c) $v_1 = 1 \quad v_2 = x \quad v_3 = x^2$

$$w_1 = v_1 \qquad w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = x - \frac{0}{2}\cdot 1 = x$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 = x^2 - \frac{2/3}{2}\cdot 1 - \frac{0}{2/3} x = x^2 - \frac{1}{3}$$

$$\langle w_3, w_3\rangle = (-2/3+2/3 \quad 0 \quad -2/9+2/5)\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} = (0 \quad 0 \quad 8/45)\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} = \frac{8}{45}$$

$$w_1^* = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}} \qquad w_2^* = \frac{w_2}{\|w_2\|} = \sqrt{\frac{3}{2}}\,x \qquad w_3^* = -\frac{\sqrt{5}}{2\sqrt{2}} + \frac{3\sqrt{5}}{2\sqrt{2}} x^2$$

$$M = \begin{pmatrix} 1/\sqrt{2} & 0 & -\sqrt{5}/2\sqrt{2} \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & 3\sqrt{5}/2\sqrt{2} \end{pmatrix} \qquad M^T B M = I$$

2) $\int_0^1 p(x)q(x)\,dx$

(a) (i) SIMM. $\int pq = \int qp$ sí

(ii) LIN. $\int \lambda p \cdot q = \lambda \int p\cdot q$ sí

(iii) LIN. $\int (p_1+p_2)q = \int p_1 q + \int p_2 q$ sí

(b) $\begin{cases} \langle 1,1\rangle = \int_0^1 dx = [x]_0^1 = 1 & \langle 1,x\rangle = \int_0^1 x\,dx = [x^2/2]_0^1 = 1/2 \\ \langle 1,x^2\rangle = \int_0^1 x^2 dx = [x^3/3]_0^1 = 1/3 & \langle x,x\rangle = \int_0^1 x^2 dx = 1/3 \\ \langle x,x^2\rangle = \int_0^1 x^3 dx = [x^4/4]_0^1 = 1/4 & \langle x^2,x^2\rangle = \int_0^1 x^4 dx = [x^5/5]_0^1 = 1/5 \end{cases}$

$$\leadsto B = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \qquad DET(B) = \frac{1}{15} + \frac{1}{24} + \frac{1}{24} - \frac{1}{27} - \frac{1}{16} - \frac{1}{20} =$$

$$= \frac{18}{120} - \frac{5+4}{80} - \frac{1}{27} = \frac{3}{20} - \frac{9}{80} - \frac{1}{27} = \frac{3}{80} - \frac{1}{27} = \frac{1}{2160}$$

SILV.(1,2,3): $DET(1)=1 \quad DET\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad DET(B) = 1/2160$

$$\overset{}{+}\overset{P}{\,}\;+\overset{P}{\,}\;+\overset{P}{\,}\;+ \qquad n_+ = 3 \quad n_- = 0 \quad n_0 = 0 \leadsto \text{DEF. POSITIVA}$$

(c) $V_1 = 1 \quad V_2 = x \quad V_3 = x^2$

$$W_1 = V_1 \qquad W_2 = V_2 - \frac{\langle V_2, W_1\rangle}{\langle W_1, W_1\rangle} W_1 = x - \frac{1/2}{1} 1 = x - \frac{1}{2}$$

$$\langle W_2, W_2\rangle = (-1/2 + 1/2 \quad -1/4 + 1/3 \quad -1/6 + 1/4)\begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} = (0 \quad 1/12 \quad 1/12)\begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{12}$$

$$\langle W_2, V_3\rangle = (0 \quad 1/12 \quad 1/12)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{12}$$

$$W_3 = V_3 - \frac{\langle V_3, W_1\rangle}{\langle W_1, W_1\rangle} W_1 - \frac{\langle V_3, W_2\rangle}{\langle W_2, W_2\rangle} W_2 = x^2 - \frac{1/3}{1}\cdot 1 - \frac{1/12}{1/12}\left(x - \frac{1}{2}\right) =$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2} = \frac{1}{6} - x + x^2$$

$$\langle W_3, W_3\rangle = (1/6 - 3/2 + 1/3 \quad 1/12 - 1 + 1/4 \quad 1/18 - 3/4 + 1/5)\cdot\begin{pmatrix} 1/6 \\ -1 \\ 1 \end{pmatrix} =$$

$$= (-1 \quad -2/3 \quad -83/180)\begin{pmatrix} 1/6 \\ -1 \\ 1 \end{pmatrix} = -\frac{1}{6} + \frac{2}{3} - \frac{83}{180} = \frac{1}{180}$$

$$W_1^* = \frac{W_1}{\|W_1\|} = 1 \qquad W_2^* = \frac{W_2}{\|W_2\|} = -\sqrt{3} + 2\sqrt{3}x \qquad W_3^* = \frac{W_3}{\|W_3\|} = 6\sqrt{5}\left(\frac{1}{6} - x + x^2\right)$$

$$M = \begin{pmatrix} 1 & -\sqrt{3} & \sqrt{5} \\ 0 & 2\sqrt{3} & -6\sqrt{5} \\ 0 & 0 & 6\sqrt{5} \end{pmatrix} \qquad M^T B M = I$$

3) $\int_{-1}^{1} p'(x)\,q'(x)\,dx$

(a) (i) SIMM. $\int p'q' = \int q'p'$ SÍ

(ii) LIN. $\int (2p)'\cdot q' = 2\int p'\cdot q'$ SÍ

(iii) LIN. $\int (p_1+p_2)' q = \int p_1' q + \int p_2' q$ SÍ

(b)
$$\begin{cases} \langle 1,1\rangle = \int_{-1}^{1} 0\,dx = 0 \quad \langle 1,x\rangle = \int_{-1}^{1} 0\,dx = 0 \quad \langle 1,x^2\rangle = \int_{-1}^{1} 0\,dx = 0 \\ \langle x,x\rangle = \int_{-1}^{1} 1\,dx = [x]_{-1}^{1} = 2 \quad \langle x,x^2\rangle = \int_{-1}^{1} 2x\,dx = [x^2]_{-1}^{1} = 0 \\ \langle x^2,x^2\rangle = \int_{-1}^{1} 4x^2\,dx = [4x^3/3]_{-1}^{1} = 8/3 \end{cases}$$

$$\rightarrow B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8/3 \end{pmatrix}$$

$n_+ = 2$ $n_0 = 1$ $n_- = 0$ $\rightarrow$ SEMIDEF. POSITIVA

4) $\int_{0}^{1} p(x)\,q'(x)\,dx$

(a) (i) SIMM. $\int p q' \neq \int q p'$ NO

(ii) LIN. $\int (2p) q' = 2\int p q'$ $\quad \int p(2q)' = 2\int p q'$ SÍ

(iii) LIN. $\int (p_1+p_2) q' = \int p_1 q + \int p_2 q$ $\quad \int p(q_1+q_2)' = \int p q_1' + \int p q_2'$ SÍ

5) $p(x)\,q(x)$ NON È $f: V\times V \to \mathbb{R}$

6) $p(0)\,q(0)$

(a) (i) SIMM. $p(0)q(0) = q(0)p(0)$ SÍ

(ii) LIN. $[2p(0)]\,q(0) = 2\,p(0)\,q(0)$ SÍ

(iii) LIN. $[p_1(0)+p_2(0)]\,q(0) = p_1(0)\,q(0) + p_2(0)\,q(0)$ SÍ

(b) $\begin{cases} \langle 1,1\rangle = 1\cdot 1 = 1 & \langle 1,x\rangle = 1\cdot 0 = 0 & \langle 1,x^2\rangle = 1\cdot 0 = 0 \\ \langle x,x\rangle = 0\cdot 0 = 0 & \langle x,x^2\rangle = 0\cdot 0 = 0 & \langle x^2,x^2\rangle = 0\cdot 0 = 0 \end{cases}$

$\rightsquigarrow B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $m_+ = 1$ $m_0 = 2$ $m_- = 0$ $\rightsquigarrow$ SEMIDEF. POSITIVA

7) $p(1)\,q(0)$

(a) (i) SIMM. $p(1)q(0) \neq q(1)\cdot p(0)$ NO

(ii) LIN. $[\lambda p(1)]\,q(0) = \lambda p(1) q(0)$ SÍ

(iii) LIN. $[p_1(1)+p_2(1)]\,q(0) = p_1(1)q(0) + p_2(1)q(0)$ SÍ

8) $p(0)q(1) + p(1)q(0)$

(a) (i) SIMM. $p(0)q(1) + p(1)q(0) = q(0)p(1) + q(1)p(0)$ SÍ

(ii) LIN. $[\lambda p(0)]q(1) + [\lambda p(1)]q(0) = \lambda[p(0)q(1) + p(1)q(0)]$ SÍ

(iii) LIN. $[p_1(0)+p_2(0)]q(1) + [p_1(1)+p_2(1)]q(0) = [p_1(0)q(1) + p_1(1)q(0)] + [p_2(0)q(1) + p_2(1)q(0)]$ SÍ

(b) $\begin{cases} \langle 1,1\rangle = 1+1 = 2 & \langle 1,x\rangle = 1+0 = 1 & \langle 1,x^2\rangle = 1+0 = 1 \\ \langle x,x\rangle = 0+0 = 0 & \langle x,x^2\rangle = 0+0 = 0 & \langle x^2,x^2\rangle = 0+0 = 0 \end{cases}$

$\rightsquigarrow B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ COMPL. QUAD.: $x^2 + 2xy + 2xz = (x+y+z)^2 - y^2 - z^2 - 2yz =$

$= (x+y+z)^2 - (y+z)^2$

$m_+ = 1$ $m_- = 1$ $m_0 = 1$ $\rightsquigarrow$ INDEFINITA

9) $p'(0)\,q'(0)$

(a) (i) SIMM. $p'(0)q'(0) = q'(0)p'(0)$ SÍ

(ii) LIN. $(\lambda p'(0))\,q'(0) = \lambda p'(0) q'(0)$ SÍ

(iii) LIN. $[p_1'(0) + p_2'(0)]\,q'(0) = p_1'(0)q(0) + p_2'(0)q(0)$ SÍ

(b) $\begin{cases} \langle 1,1\rangle = 0 & \langle 1,x\rangle = 0 & \langle 1,x^2\rangle = 0 \\ \langle x,x\rangle = 1 & \langle x,x^2\rangle = 0 & \langle x^2,x^2\rangle = 0 \end{cases}$ $\rightsquigarrow B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $m_+ = 1$ $m_0 = 2$ $m_- = 0$ $\rightsquigarrow$ SEMID. POS.

10) $p(0)q(0) + p(1)q(1) + p(-1)q(-1)$

(a) (i) SIMM. $\sum p(a)q(a) = \sum q(a)p(a)$ SÍ

(ii) LIN. $\sum [\lambda p(a)]q(a) = \lambda \sum q(a)p(a)$ SÍ

(iii) LIN. $\sum [p_1(a)+p_2(a)]q(a) = \sum p_1(a)q(a) + \sum p_2(a)q(a)$ SÍ

(b)
$$\begin{cases} \langle 1,1\rangle = 1+1+1 = 3 & \langle 1,x\rangle = 0+1-1 = 0 & \langle 1,x^2\rangle = 0+1+1 = 2 \\ \langle x,x\rangle = 0+1+1 = 2 & \langle x,x^2\rangle = 0+1-1 = 0 & \langle x^2,x^2\rangle = 0+1+1 = 2 \end{cases}$$

$$\rightsquigarrow B = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \qquad DET(B) = 12-8 = 4$$

SYLVESTER(1,2,3): $DET(3) = 3 \quad DET\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 6 \quad DET(B) = 4$

$$\overset{P\ P\ P}{+ + + +} \qquad n_+ = 3 \quad n_0 = 0 \quad n_- = 0 \rightsquigarrow \text{DEF. POSITIVA}$$

(c) $v_1 = 1 \quad v_2 = x \quad v_3 = x^2$

$$w_1 = v_1 \qquad w_2 = v_2 - \frac{\langle v_2,w_1\rangle}{\langle w_1,w_1\rangle} w_1 = x - \frac{0}{3}\cdot 1 = x$$

$$w_3 = v_3 - \frac{\langle v_3,w_1\rangle}{\langle w_1,w_1\rangle} w_1 - \frac{\langle v_3,w_2\rangle}{\langle w_2,w_2\rangle} w_2 = x^2 - \frac{2}{3}1 - \frac{0}{2}x = -\frac{2}{3} + x^2$$

$$\langle w_3,w_3\rangle = \begin{pmatrix} -2+2 & 0 & -4/3+2 \end{pmatrix} \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix} = \frac{2}{3}$$

$$w_1^* = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{3}} \qquad w_2^* = \frac{w_2}{\|w_2\|} = \frac{x}{\sqrt{2}} \qquad w_3^* = \sqrt{\frac{3}{2}}\left(-\frac{2}{3} + x^2\right)$$

$$M = \begin{pmatrix} 1/\sqrt{3} & 0 & -\frac{2}{3}\sqrt{\frac{3}{2}} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} \end{pmatrix} \qquad M^T B M = I$$

11) $p(0)q(0) + p'(1)q'(1) + p(-1)q(-1)$

(a) (i) SIMM. SÍ (ii) LIN. SÍ (iii) LIN. SÍ

(b)
$$\begin{cases} \langle 1,1\rangle = 1+0+1 = 2 & \langle 1,x\rangle = 0+0-1 = -1 & \langle 1,x^2\rangle = 0+0+1 = 1 \\ \langle x,x\rangle = 0+1+1 = 2 & \langle x,x^2\rangle = 0+2-1 = 1 & \langle x^2,x^2\rangle = 0+4+1 = 5 \end{cases}$$

$$\rightsquigarrow B = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 5 \end{pmatrix} \qquad DET(B) = 20 - 1 - 1 - 2 - 2 - 5 = 9$$

$$SILV.(1,2,3): \quad DET(2) = 2 \quad DET\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3 \quad DET(B) = 9$$

$$+\overset{P}{}+\overset{P}{}+\overset{P}{}+ \qquad m_+ = 3 \quad m_- = 0 \quad m_0 = 0 \rightsquigarrow \text{DEF. POSITIVA}$$

(c) $v_1 = 1 \quad v_2 = x \quad v_3 = x^2$

$$w_1 = v_1 \qquad w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = x - \frac{-1}{2} 1 = \frac{1}{2} + x$$

$$\langle w_2, w_2 \rangle = (0 \quad 3/2 \quad 3/2) \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} = \frac{3}{2}$$

$$\langle w_2, v_3 \rangle = (0 \quad 3/2 \quad 3/2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{3}{2}$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 = x^2 - \frac{1}{2} 1 - \frac{3/2}{3/2}\left(\frac{1}{2} + x\right) = -1 - x + x^2$$

$$\langle w_3, w_3 \rangle = (0 \quad 0 \quad 3) \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 3$$

$$w_1^* = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}} \qquad w_2^* = \frac{w_2}{\|w_2\|} = \sqrt{\frac{2}{3}}\left(\frac{1}{2} + x\right) \qquad w_3^* = \frac{1}{\sqrt{3}}\left(-1 - x + x^2\right)$$

$$M = \begin{pmatrix} 1/\sqrt{2} & \sqrt{2}/2\sqrt{3} & -1/\sqrt{3} \\ 0 & \sqrt{2/3} & -1/\sqrt{3} \\ 0 & 0 & 1/\sqrt{3} \end{pmatrix} \qquad M^{\delta} B M = I$$

12) $(p(0) + 2p(1))(q(0) + 2q(1))$

(a) (i) SIMM. $(q(0)+2q(1))(p(0)+2p(1)) = (p(0)+2p(1))(q(0)+2q(1))$ SI

(ii) LIN. $(\lambda p(0) + 2\lambda p(1))(q(0)+2q(1)) = \lambda (p(0)+2p(1))(q(0)+2q(1))$ SI

(iii) LIN. $(p_1(0)+p_2(0)+2p_1(1)+2p_2(1)) = (p_1(0)+2p_1(1))(q(0)+2q(1)) + (p_2(0)+2p_2(1))(q(0)+2q(1))$ SI

(b) $\begin{cases} \langle 1,1 \rangle = 3\cdot 3 = 9 \quad \langle 1,x \rangle = 3 \cdot 2 = 6 \quad \langle 1, x^2 \rangle = 3 \cdot 2 = 6 \\ \langle x,x \rangle = 2 \cdot 2 = 4 \quad \langle x, x^2 \rangle = 2 \cdot 2 = 4 \quad \langle x^2, x^2 \rangle = 2 \cdot 2 = 4 \end{cases}$

$$\leadsto B=\begin{pmatrix} 9 & 6 & 6 \\ 6 & 5 & 5 \\ 6 & 5 & 5 \end{pmatrix} \leadsto \text{RANGO } 1 \quad n_0=2$$

$$\text{C.Q.}: 9x^2+5y^2+5z^2+12xy+12xz+8yz = (3x+2y+2z)^2$$

$$n_+=1 \quad n_0=2 \quad n_-=0 \leadsto \text{SEMIDEF. POSITIVA}$$

13) $(p(0)+2q(1))(q(0)+2p(1))$

(a) (i) SIMM. $(q(0)+2p(1))(p(0)+2q(1))=(p(0)+2q(1))(q(0)+2p(1))$ SÍ

(ii) LIN. $(\lambda p(0)+2q(1))(q(0)+2\lambda p(1)) \neq \lambda(p(0)+2q(1))(q(0)+2p(1))$ NO

(iii) LIN. $(p_1(0)+p_2(0)+2q(1))(q(0)+2p_1(1)+2p_2(2)) =$

$=(p_1(0)+2q(1))(q(0)+2p_1(1)) + p_2(0)(q(0)+2p_1(1)) +$

$+(p_1(0)+2q(1))(2p_2(2)) + p_2(0)\cdot 2p_2(2)$ NO

15) $p(0)q(0)-p(1)q(1)$

(a) (i) SIMM. $q(0)p(0)-q(1)p(1) = p(0)q(0)-p(1)q(1)$ SÍ

(ii) LIN. $\lambda p(0)q(0)-\lambda p(1)q(1) = \lambda[p(0)q(0)-p(1)q(1)]$ SÍ

(iii) LIN. $[p_1(0)+p_2(0)]q(0)-[p_1(1)+p_2(1)]q(1)=[p_1(0)q(0)-p_1(1)q(1)]+[p_2(0)q(0)-p_2(1)q(1)]$ SÍ

(b) $\begin{cases} \langle 1,1\rangle = 1-1=0 \quad \langle 1,x\rangle = 0-1=-1 \quad \langle 1,x^2\rangle = 0-1=-1 \\ \langle x,x\rangle = 0-1=-1 \quad \langle x,x^2\rangle = 0-1=-1 \quad \langle x^2,x^2\rangle = 0-1=-1 \end{cases}$

$$\leadsto B=\begin{pmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$\text{C.Q.}: -y^2-z^2-2xy-2xz-2yz = -(x+y+z)^2+x^2$$

$$n_+=1 \quad n_-=1 \quad n_0=1 \leadsto \text{INDEFINITA}$$

13) $\int_{-1}^{1} p'(x)q'(x)\,dx + p(3)q(3)$

(a) (i) SIMM. $\int q'p' + q(3)p(3) = \int p'q' + p(3)q(3)$ SÍ

(ii) LIN. $\int \lambda p'q' + \lambda p(3)q(3) = \lambda[\int p'q' + p(3)q(3)]$ SÍ

(iii) LIN. $\int (p_1'+p_2')q' + [p_1(3)+p_2(3)]q(3) = [\int p_1'q' + p_1(3)q(3)] + [\int p_2'q' + p_2(3)q(3)]$ SÍ

(b) $\begin{cases} \langle 1,1\rangle = 0+1\cdot 1 = 1 \quad \langle 1,x\rangle = 0+1\cdot 3 = 3 \quad \langle 1,x^2\rangle = 0+1\cdot 9 = 9 \\ \langle x,x\rangle = \int_{-1}^{1} dx + 3\cdot 3 = 2+9 = 11 \quad \langle x,x^2\rangle = \int_{-1}^{1} 2x\,dx + 3\cdot 9 = 0+27 = 27 \\ \langle x^2,x^2\rangle = \int_{-1}^{1} 5x^2\,dx + 9\cdot 9 = [5/3\,x^3]_{-1}^{1} + 81 = \frac{8}{3}+81 = \frac{251}{3} \end{cases}$

$$\rightsquigarrow B = \begin{pmatrix} 1 & 3 & 9 \\ 3 & 11 & 27 \\ 9 & 27 & 251/3 \end{pmatrix} \qquad DET(B) = \frac{2761}{3} + \cancel{729} + 729 - 891 - \cancel{729} - 753 =$$
$$= \frac{2761}{3} - 915 = \frac{2761-2745}{3} = \frac{16}{3}$$

SYLVESTER (1,2,3): $DET(1) = 1 \quad DET\begin{pmatrix} 1 & 3 \\ 3 & 11 \end{pmatrix} = 2 \quad DET(B) = 16/3$

$$\overset{P}{+}\overset{P}{+}\overset{P}{+}+ \quad n_+ = 3 \quad n_- = 0 \quad n_0 = 0 \rightsquigarrow \text{DEF. POSITIVA}$$

(c) $v_1 = 1 \quad v_2 = x \quad v_3 = x^2$

$$w_1 = v_1 \qquad w_2 = v_2 - \frac{\langle v_2,w_1\rangle}{\langle w_1,w_1\rangle} w_1 = x - \frac{3}{1}1 = -3+x$$

$$\langle w_2,w_2\rangle = (0\ \ 2\ \ 0)\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = 2 \qquad \langle w_2,v_3\rangle = (0\ 2\ 0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$w_3 = v_3 - \frac{\langle v_3,w_1\rangle}{\langle w_1,w_1\rangle} w_1 - \frac{\langle v_3,w_2\rangle}{\langle w_2,w_2\rangle} w_2 = x^2 - \frac{9}{1}1 - \frac{0}{2}(-3+x) = -9+x^2$$

$$\langle w_3,w_3\rangle = \left(0\ \ 0\ \ -81+\frac{251}{3}\right)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{251-243}{3} = \frac{8}{3}$$

$$w_1^* = \frac{w_1}{\|w_1\|} = 1 \qquad w_2^* = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{2}}(-3+x) \qquad w_3^* = \frac{\sqrt{3}}{2\sqrt{2}}(-9+x^2)$$

$$M = \begin{pmatrix} 1 & -3/\sqrt{2} & -9\sqrt{3}/2\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & \sqrt{3}/2\sqrt{2} \end{pmatrix} \qquad M^T B M = I$$

16) $\int_{-2}^{2} p(x)q(x)\,dx - p(2)q(2)$

(a) (i) SIMM. $\int q\,p - q(2)p(2) = \int p\,q - p(2)q(2)$ SÌ

(ii) LIN. $\int \lambda p\,q - \lambda p(2)q(2) = \lambda\left[\int p\,q - p(2)q(2)\right]$ SÌ

(iii) LIN. $\int (p_1+p_2)q - [p_1(2)+p_2(2)]q(2) = \left[\int p_1 q - p_1(2)q(2)\right] + \left[\int p_2 q - p_2(2)q(2)\right]$ SÌ