Prodotti scalari 1

Argomenti: prodotti scalari generali **Difficoltà**: ⋆⋆⋆

Prerequisiti: prodotti scalari, forme quadratiche, Gram-Schmidt

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1. Consideriamo i prodotti scalari in $\mathbb{R}^2$ rappresentati, rispetto alla base canonica, dalle seguenti matrici:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \qquad \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Per ciascuno di essi si richiede di

(a) determinare se è definito positivo oppure no,
(b) determinare il prodotto scalare tra i vettori $(1, 2)$ e $(3, -1)$,
(c) determinare la matrice che lo rappresenta rispetto alla base $\{(-1, 2), (3, -2)\}$ (si consiglia per le prime volte di svolgere questo punto sia direttamente con la definizione, sia con il cambio di base),
(d) determinare l'equazione cartesiana del sottospazio ortogonale al vettore $(-1, 1)$,
(e) determinare un vettore ortogonale al sottospazio di equazione cartesiana $x+2y = 0$,
(f) determinare una base ortonormale di $\mathbb{R}^2$ (limitatamente a quelli definiti positivi).

2. Consideriamo i prodotti scalari in $\mathbb{R}^3$ rappresentati, rispetto alla base canonica, dalle seguenti matrici:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 6 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

Per ciascuno di essi si richiede di

(a) verificare che è definito positivo,
(b) determinare la matrice che lo rappresenta rispetto alla base

$$\{(-1, 2, 0), (3, 0, -2), (1, 1, 1)\},$$

(c) determinare l'equazione cartesiana del sottospazio ortogonale al vettore $(-1, 1, 3)$,
(d) determinare una base ortogonale, costituita da vettori a coordinate intere, del sottospazio di equazione cartesiana $x = 3y - z$,
(e) determinare un vettore a coordinate intere ortogonale a $(1, 0, 0)$ e a $(0, 1, 1)$,
(f) determinare la proiezione ortogonale del vettore $(1, 0, 0)$ sul sottospazio generato da $(0, 1, 0)$ e $(0, 0, 1)$,
(g) determinare una base ortonormale di $\mathbb{R}^3$,
(h) determinare la matrice che, rispetto alla base canonica, rappresenta la proiezione sul sottospazio di equazione cartesiana $y + z = 0$.

SOLUZIONI?
BY GIMUSI 25/01/15

1. Consideriamo i prodotti scalari in $\mathbb{R}^2$ rappresentati, rispetto alla base canonica, dalle seguenti matrici:

$$1)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad 2)\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \quad 3)\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \quad 5)\begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} \quad 5)\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Per ciascuno di essi si richiede di

(a) determinare se è definito positivo oppure no,

(b) determinare il prodotto scalare tra i vettori $(1, 2)$ e $(3, -1)$,

(c) determinare la matrice che lo rappresenta rispetto alla base $\{(-1, 2), (3, -2)\}$ (si consiglia per le prime volte di svolgere questo punto sia direttamente con la definizione, sia con il cambio di base),

(d) determinare l'equazione cartesiana del sottospazio ortogonale al vettore $(-1, 1)$,

(e) determinare un vettore ortogonale al sottospazio di equazione cartesiana $x + 2y = 0$,

(f) determinare una base ortonormale di $\mathbb{R}^2$ (limitatamente a quelli definiti positivi).

1) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(a) $DET(B) = 2 > 0 \quad 1 > 0 \rightsquigarrow$ DEF. POSITIVO $(x^T B x > 0 \ \forall x \neq 0)$

(b) $\langle v_1, v_2 \rangle_B = v_1^T B v_2 = (1\ 2)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = (1\ 5)\begin{pmatrix} 3 \\ -2 \end{pmatrix} = 3 - 5 = -1$

(c) $\{w_1, w_2\} \quad w_1 = (-1, 2) \quad w_2 = (3, -2)$

MODO 1 $\langle w_1, w_1 \rangle_B = (-1, 2)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = (-1\ 5)\begin{pmatrix} -1 \\ 2 \end{pmatrix} = 9$

$\langle w_2, w_2 \rangle_B = (3, -2)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = (3\ -5)\begin{pmatrix} 3 \\ -2 \end{pmatrix} = 17$

$\langle w_1, w_2 \rangle_B = (-1, 2)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = (-1\ 5)\begin{pmatrix} 3 \\ -2 \end{pmatrix} = -11 = \langle w_2, w_1 \rangle_B$

$\rightsquigarrow B_\varepsilon = \begin{pmatrix} 9 & -11 \\ -11 & 17 \end{pmatrix}$

MODO 2 $M = \begin{array}{c} \begin{array}{cc} w_1 & w_2 \end{array} \\ \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} \\ \varepsilon \to e \end{array}$ $\quad DET(M) = 2 - 6 = -4 \qquad v_e = M v_\varepsilon \ , \ v_{e,1}^T B v_{e,2}$

$\rightsquigarrow v_\varepsilon^T M^T B M v_\varepsilon \rightsquigarrow B_\varepsilon = M^T B M = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} =$

$= \begin{pmatrix} -1 & 4 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 9 & -11 \\ -11 & 17 \end{pmatrix}$

(d) $<(x,y),(-2,1)>_B = (x\ y)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}-2\\ 1\end{pmatrix} = (x+2y)\begin{pmatrix}-2\\ 2\end{pmatrix} = 0$

$\rightsquigarrow x-2y=0$

(e) $x+2y=0 \rightsquigarrow V=(-2\delta,\delta)$ $\quad <(x,y),(-2\delta,\delta)>_B = (x,y)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}-2\delta\\ \delta\end{pmatrix} =$

$= (x\ 2y)\begin{pmatrix}-2\delta\\ \delta\end{pmatrix} = -2\delta x + 2\delta y = 0 \rightsquigarrow x=y \quad w=(\delta,\delta)$

ex $(-2\ 2)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = (-2\ 2)\begin{pmatrix}1\\ 2\end{pmatrix} = -2+2=0$

(f) $V_1=(1,0) \qquad V_2=(0,1)$

$W_1 = V_1$

$<V_2,V_1>_B = (1\ 0)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = (1\ 2)\begin{pmatrix}0\\ 1\end{pmatrix} = 2$

$<V_1,V_1>_B = (1\ 0)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = (1\ 2)\begin{pmatrix}1\\ 0\end{pmatrix} = 1$

$W_2 = V_2 - \frac{<V_2,V_1>_B}{<V_1,V_1>_B} V_1 = (0,1) - 2(1,0) = (-2,1)$

$<W_1,W_2>_B = (1,0)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}-2\\ 1\end{pmatrix} = (1,2)\begin{pmatrix}-2\\ 1\end{pmatrix} = 0$

$<W_2,W_2>_B = (-2\ 1)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}-2\\ 1\end{pmatrix} = (-2,2)\begin{pmatrix}-2\\ 1\end{pmatrix} = 6$

$\rightsquigarrow W_1^* = \frac{W_1}{\|W_1\|_B} = (1,0) \qquad W_2^* = \frac{W_2}{\|W_2\|_B} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$<W_2^*,W_2^*>_B = \left(\frac{-2}{\sqrt{6}}\ \frac{1}{\sqrt{6}}\right)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}-2/\sqrt{6}\\ 1/\sqrt{6}\end{pmatrix} = \left(\frac{-2}{\sqrt{6}}\ \frac{2}{\sqrt{6}}\right)\begin{pmatrix}-2/\sqrt{6}\\ 1/\sqrt{6}\end{pmatrix} = \frac{4}{6}+\frac{2}{6} = 1$

2) $\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}$

(a) $DET(B) = -2 < 0 \quad \lambda_1 = 1, \lambda_2 = -2 \quad \rightsquigarrow$ INDEFINITA

(b) $(1\ 2)\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}3\\ -1\end{pmatrix} = (1\ -4)\begin{pmatrix}3\\ -1\end{pmatrix} = 7$

(c) $\{w_1, w_2\} \quad w_1 = (-1, 2) \quad w_2 = (3, -2)$

Modo 1 $\langle w_1, w_1\rangle_B = (-1, 2)\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}-1\\ 2\end{pmatrix} = (-1\ -4)\begin{pmatrix}-1\\ 2\end{pmatrix} = -7$

$\langle w_2, w_2\rangle_B = (3, -2)\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}3\\ -2\end{pmatrix} = (3\ \ 4)\begin{pmatrix}3\\ -2\end{pmatrix} = 1$

$\langle w_1, w_2\rangle_B = (-1, 2)\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}3\\ -2\end{pmatrix} = (-1\ -4)\begin{pmatrix}3\\ -2\end{pmatrix} = 5 \quad = \langle w_2, w_1\rangle_B$

$\rightsquigarrow B_\varepsilon = \begin{pmatrix}-7 & 5\\ 5 & 1\end{pmatrix}$

Modo 2 $M = \begin{array}{c} \begin{matrix} w_1 & w_2 \end{matrix} \\ \begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix} \\ \varepsilon \to e \end{array} \quad DET(M) = 2 - 6 = -4 \quad v_e = M v_\varepsilon \ , \ v_{e,1}^T B v_{e,2}$

$\rightsquigarrow v_\varepsilon^T M^T B M v_\varepsilon \rightsquigarrow B_\varepsilon = M^T B M = \begin{pmatrix}-1 & 2\\ 3 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix} =$

$= \begin{pmatrix}-1 & -4\\ 3 & 4\end{pmatrix}\begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix} = \begin{pmatrix}-7 & 5\\ 5 & 1\end{pmatrix}$

(d) $(x, y)\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = (x, y)\begin{pmatrix}-1\\ -2\end{pmatrix} = -x - 2y = 0 \rightsquigarrow x + 2y = 0$

(e) $x + 2y = 0 \rightsquigarrow v = (-2\delta, \delta) \quad \langle (x, y), (-2\delta, \delta)\rangle_B = (x, y)\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}-2\delta\\ \delta\end{pmatrix} =$

$= (x\ -2y)\begin{pmatrix}-2\delta\\ \delta\end{pmatrix} = -2\delta x - 2\delta y = 0 \rightsquigarrow x = -y \quad w = (\delta, -\delta)$

ex $(-2\ \ 1)\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} = (-2\ -2)\begin{pmatrix}1\\ -1\end{pmatrix} = -2 + 2 = 0$

3) $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$

(a) $DET(B) = 10-9 = 1 > 0 \quad 2>0 \rightsquigarrow$ DEF. POSITIVA

(b) $(1 \; 2)\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = (8 \; 13)\begin{pmatrix} 3 \\ -1 \end{pmatrix} = 11$

(c) $\{w_1, w_2\} \quad w_1 = (-1, 2) \quad w_2 = (3, -2)$

Modo 1 $\langle w_1, w_1 \rangle_B = (-1, 2)\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = (4 \; 7)\begin{pmatrix} -1 \\ 2 \end{pmatrix} = 10$

$\langle w_2, w_2 \rangle_B = (3, -2)\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = (0 \; -1)\begin{pmatrix} 3 \\ -2 \end{pmatrix} = 2$

$\langle w_1, w_2 \rangle_B = (-1, 2)\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = (4 \; 7)\begin{pmatrix} 3 \\ -2 \end{pmatrix} = -2 \quad = \langle w_2, w_1 \rangle_B$

$\rightsquigarrow B_\varepsilon = \begin{pmatrix} 10 & -2 \\ -2 & 2 \end{pmatrix}$

Modo 2 $M = \underset{\varepsilon \to e}{\begin{array}{cc} w_1 & w_2 \\ \left(\begin{array}{c} -1 \\ 2 \end{array}\right. & \left.\begin{array}{c} 3 \\ -2 \end{array}\right) \end{array}}$ $\quad DET(M) = 2-6 = -4 \qquad v_e = M v_\varepsilon \; , \; v_{e,1}^T B \, v_{e,2}$

$\rightsquigarrow v_\varepsilon^T M^T B M v_\varepsilon \rightsquigarrow B_\varepsilon = M^T B M = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} =$

$= \begin{pmatrix} 4 & 7 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 10 & -2 \\ -2 & 2 \end{pmatrix}$

(d) $(x, y)\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = (x, y)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = x + 2y = 0$

(e) $x + 2y = 0 \rightsquigarrow v = (-2\delta, \delta) \quad \langle (x, y), (-2\delta, \delta) \rangle_B = (x, y)\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -2\delta \\ \delta \end{pmatrix} =$

$= (2x+3y \;\; 3x+5y)\begin{pmatrix} -2\delta \\ \delta \end{pmatrix} = (-4x-6y+3x+5y)\delta = 0 \quad x = -y \quad w = (\delta, -\delta)$

ex $(-2 \; 1)\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1 \; -1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 + 1 = 0$

(1) $v_1=(1,0) \quad v_2=(0,1)$

$w_1 = v_1$

$\langle v_2, v_2\rangle_B = (1,0)\begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = (2,3)\begin{pmatrix}0\\1\end{pmatrix} = 3$

$\langle v_1, v_1\rangle_B = (1,0)\begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = (2,3)\begin{pmatrix}1\\0\end{pmatrix} = 2$

$w_2 = v_2 - \frac{\langle v_2, v_1\rangle_B}{\langle v_1, v_1\rangle_B} v_1 = (0,1) - \frac{3}{2}(1,0) = \left(-\frac{3}{2}, 1\right)$

$\langle w_1, w_2\rangle_B = (1\ 0)\begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}-3/2\\1\end{pmatrix} = (2\ 3)\begin{pmatrix}-3/2\\1\end{pmatrix} = 0$

$\langle w_2, w_2\rangle_B = (-3/2\ \ 1)\begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}-3/2\\1\end{pmatrix} = (0\ \ 1/2)\begin{pmatrix}-3/2\\1\end{pmatrix} = 1/2$

$\rightsquigarrow w_1^* = \frac{w_1}{\|w_1\|_B} = \left(\frac{\sqrt{2}}{2}, 0\right) \qquad w_2^* = \frac{w_2}{\|w_2\|_B} = \left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right)$

$\langle w_2^*, w_2^*\rangle_B = \left(-\frac{3\sqrt{2}}{2}\ \ \sqrt{2}\right)\begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}-\frac{3\sqrt{2}}{2}\\ \sqrt{2}\end{pmatrix} = \left(0\ \ \sqrt{2}/2\right)\begin{pmatrix}-\frac{3\sqrt{2}}{2}\\ \sqrt{2}\end{pmatrix} = 1$

5) $\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}$

(a) $DET(B) = 8 - 9 = -1 \rightsquigarrow$ INDEFINITA

(b) $(1,2)\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}3\\-1\end{pmatrix} = (-4\ \ 7)\begin{pmatrix}3\\-1\end{pmatrix} = -17$

(c) $\{w_1, w_2\}$ $w_1 = (-1,2)$ $w_2 = (3,-2)$

Modo 1 $\langle w_1, w_1\rangle_B = (-1,2)\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = (-8\ \ 11)\begin{pmatrix}-1\\2\end{pmatrix} = 30$

$\langle w_2, w_2\rangle_B = (3,-2)\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = (12\ -17)\begin{pmatrix}3\\-2\end{pmatrix} = 70$

$$\langle w_1, w_2\rangle_B = (-1,2)\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}3\\ -2\end{pmatrix} = (-8\ 11)\begin{pmatrix}3\\ -2\end{pmatrix} = -56 = \langle w_2, w_1\rangle_B$$

$$\rightsquigarrow B_\mathcal{E} = \begin{pmatrix}30 & -56\\ -56 & 70\end{pmatrix}$$

Modo 2 $\quad M = \begin{array}{cc} w_1 & w_2 \\ \end{array}\begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix}_{\mathcal{E}\to e}$ $\quad DET(M) = 2-6 = -5 \qquad v_e = M v_\mathcal{E}\ ,\ v_{e,1}^T B v_{e,2}$

$$\rightsquigarrow v_\mathcal{E}^T M^T B M v_\mathcal{E} \rightsquigarrow B_\mathcal{E} = M^T B M = \begin{pmatrix}-1 & 2\\ 3 & -2\end{pmatrix}\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix} =$$

$$= \begin{pmatrix}-8 & 11\\ 12 & -19\end{pmatrix}\begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix} = \begin{pmatrix}30 & -56\\ -56 & 70\end{pmatrix}$$

(d) $(x,y)\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = (x,y)\begin{pmatrix}-5\\ 8\end{pmatrix} = -5x+8y=0$

(e) $x+2y=0 \rightsquigarrow v = (-2\delta, \delta) \quad \langle (x,y),(-2\delta,\delta)\rangle_B = (x,y)\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}-2\delta\\ \delta\end{pmatrix} =$

$= (2x-3y\ \ -3x+5y)\begin{pmatrix}-2\delta\\ \delta\end{pmatrix} = (-4x+6y-3x+5y)\,\delta = 0 \quad -7x+11y=0 \quad w = (11\delta, 7\delta)$

(f) B INDEFINITA

5) $B = \begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}$

(a) $|B-\lambda I| = \begin{vmatrix}1-\lambda & 2\\ 2 & 5-\lambda\end{vmatrix} = \lambda^2 - 5\lambda \quad n_0 = 1 \quad n_+ = 1$ SEMIDEF. POS.

(b) $(1\ \ 2)\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}3\\ -1\end{pmatrix} = (5\ \ 10)\begin{pmatrix}3\\ -1\end{pmatrix} = 5$

(c) $\{w_1, w_2\} \quad w_1 = (-1,2) \quad w_2 = (3,-2)$

Modo 1 $\quad \langle w_1, w_1\rangle_B = (-1,2)\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}-1\\ 2\end{pmatrix} = (3\ 8)\begin{pmatrix}-1\\ 2\end{pmatrix} = 13$

$$\langle w_2, w_2\rangle_B = (3,-2)\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}3\\ -2\end{pmatrix} = (-1\ -4)\begin{pmatrix}3\\ -2\end{pmatrix} = 5$$

$$\langle w_1, w_2\rangle_B = (-1,2)\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}3\\ -2\end{pmatrix} = (3\ 6)\begin{pmatrix}3\\ -2\end{pmatrix} = -3 \quad = \langle w_2, w_1\rangle_B$$

$$\rightsquigarrow B_\varepsilon = \begin{pmatrix}9 & -3\\ -3 & 1\end{pmatrix}$$

Modo 2

$$M = \begin{array}{c} \begin{array}{cc} w_1 & w_2 \end{array} \\ \begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix} \\ \varepsilon \to e \end{array} \qquad \det(M) = 2-6 = -5 \qquad v_e = M v_\varepsilon, \quad v_{e,1}^T B\, v_{e,2}$$

$$\rightsquigarrow v_\varepsilon^T M^T B M v_\varepsilon \rightsquigarrow B_\varepsilon = M^T B M = \begin{pmatrix}-1 & 2\\ 3 & -2\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix} =$$

$$= \begin{pmatrix}3 & 6\\ -1 & -2\end{pmatrix}\begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix} = \begin{pmatrix}9 & -3\\ -3 & 1\end{pmatrix}$$

(d) $(x,y)\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = (x,y)\begin{pmatrix}1\\ 2\end{pmatrix} = 0 \qquad x+2y=0$

(e) $x+2y=0 \rightsquigarrow V = (-2\delta, \delta) \quad \langle (x,y), (-2\delta,\delta)\rangle_B = (x,y)\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}-2\delta\\ \delta\end{pmatrix} =$

$= (x+2y \quad 2x+5y)\begin{pmatrix}-2\delta\\ \delta\end{pmatrix} = (-2x-5y+2x+5y)\,\delta = 0 \qquad V = (s,\delta) \quad s,\delta \in \mathbb{R}$

(f) B SEMIDEFINITA POSITIVA

2. Consideriamo i prodotti scalari in $\mathbb{R}^3$ rappresentati, rispetto alla base canonica, dalle seguenti matrici:

$$1)\begin{pmatrix}1&1&0\\1&1&1\\0&1&3\end{pmatrix}\quad 2)\begin{pmatrix}1&-1&0\\-1&3&1\\0&1&1\end{pmatrix}\quad 3)\begin{pmatrix}1&1&1\\1&2&1\\1&1&3\end{pmatrix}\quad 5)\begin{pmatrix}6&1&0\\1&1&2\\0&2&5\end{pmatrix}$$

Per ciascuno di essi si richiede di

(a) verificare che è definito positivo,

(b) determinare la matrice che lo rappresenta rispetto alla base

$$\{(-1,2,0),(3,0,-2),(1,1,1)\},$$

(c) determinare l'equazione cartesiana del sottospazio ortogonale al vettore $(-1,1,3)$,

(d) determinare una base ortogonale, costituita da vettori a coordinate intere, del sottospazio di equazione cartesiana $x=3y-z$,

(e) determinare un vettore a coordinate intere ortogonale a $(1,0,0)$ e a $(0,1,1)$,

(f) determinare la proiezione ortogonale del vettore $(1,0,0)$ sul sottospazio generato da $(0,1,0)$ e $(0,0,1)$,

(g) determinare una base ortonormale di $\mathbb{R}^3$,

(h) determinare la matrice che, rispetto alla base canonica, rappresenta la proiezione sul sottospazio di equazione cartesiana $y+z=0$.

1) $$B=\begin{pmatrix}1&1&0\\1&1&1\\0&1&3\end{pmatrix}$$

(a) SYLVESTER $(3,2,1)$: $DET(1)=1$ $\quad DET\begin{pmatrix}1&1\\1&3\end{pmatrix}=2$ $\quad DET(B)=2-3=-1$

$+\overset{P}{}+\overset{P}{}+\overset{V}{}-$ $\quad n_+=2 \quad n_-=1 \rightsquigarrow$ INDEFINITA

(b)
$$B_\varepsilon=\overset{M^T}{\begin{pmatrix}-1&2&0\\3&0&-2\\1&1&1\end{pmatrix}}\overset{B}{\begin{pmatrix}1&1&0\\1&1&1\\0&1&3\end{pmatrix}}\overset{M}{\begin{pmatrix}-1&3&1\\2&0&1\\0&-2&1\end{pmatrix}}=$$

$$=\begin{pmatrix}1&1&2\\3&1&-6\\2&3&5\end{pmatrix}\begin{pmatrix}-1&3&1\\2&0&1\\0&-2&1\end{pmatrix}=\begin{pmatrix}1&-1&5\\-1&21&-2\\5&-2&9\end{pmatrix}$$

(c)
$$x^T B v=(x\ y\ z)\begin{pmatrix}1&1&0\\1&1&1\\0&1&3\end{pmatrix}\begin{pmatrix}-1\\1\\3\end{pmatrix}=(x\ y\ z)\begin{pmatrix}0\\3\\10\end{pmatrix}=0$$

$\rightsquigarrow \quad 3y+10z=0$

(d) $x = 3y - z$ BASE: $v_1 = (1, 0, -1)$ $v_2 = (3, 1, 0)$

$$\langle v_1, v_1 \rangle = v_1^\delta B v_1 = (1\ 0\ -1)\begin{pmatrix}1\\0\\-3\end{pmatrix} = 5 \qquad \langle v_1, v_2 \rangle = v_2^\delta B v_1 = 3$$

$$w_1 = v_1 \qquad \hat{w}_2 = v_2 - \frac{\langle v_1, v_2\rangle}{\langle v_1, v_1\rangle} v_1 = (3,1,0) - \frac{3}{5}(1,0,-1) = \left(\frac{9}{5}, 1, \frac{3}{5}\right)$$

BASE ORTOGONALE: $w_1 = (1, 0, -1)$ $w_2 = (9, 5, 3)$

BASE ORTONORMALE:

$$\langle w_2, w_2 \rangle = w_2^\delta B w_2 = (9\ 5\ 3)\begin{pmatrix}13\\16\\13\end{pmatrix} = 117 + 65 + 39 = 220$$

$$w_1^* = \frac{w_1}{\|w_1\|} = (1/2, 0, -1/2) \qquad w_2^* = \frac{w_2}{\|w_2\|} = \left(\frac{9}{\sqrt{220}}, \frac{5}{\sqrt{220}}, \frac{3}{\sqrt{220}}\right)$$

(e) $v_1 = (1, 0, 0)$ $v_2 = (0, 1, 1)$

$$x^\delta B v_1 = (x\ y\ z)\begin{pmatrix}1\\1\\0\end{pmatrix} = x + y = 0 \qquad x^\delta B v_2 = (x\ y\ z)\begin{pmatrix}1\\2\\5\end{pmatrix} = x + 2y + 5z = 0$$

$$\begin{cases} x + y = 0 \\ x + 2y + 5z = 0 \end{cases} \rightsquigarrow \begin{cases} x = \delta \quad y = -\delta \\ z = \delta/5 \end{cases} \rightsquigarrow (5, -5, 1)$$

(f) $w = (1, 0, 0) = w_\perp + w_\parallel$ $w_\parallel = (0, a, b) \in SPAN\{\overset{v_1}{(0,1,0)}, \overset{v_2}{(0,0,1)}\}$

$$w_\perp = w - w_\parallel = (1, -a, -b)$$

$$\begin{cases} w_\perp^\delta B v_1 = 0 \\ w_\perp^\delta B v_2 = 0 \end{cases} \quad \begin{cases} 1 - a - b = 0 \\ -a - 3b = 0 \end{cases} \quad \begin{cases} 1 + 2b = 0 \quad b = -1/2 \\ a = 3/2 \end{cases}$$

$\rightsquigarrow w_\parallel = (0, 3/2, -1/2)$

(g) $v_1 = (1, 0, 0)$ $v_2 = (0, 1, 0)$ $v_3 = (0, 0, 1)$

$$w_1 = v_1 = (1, 0, 0) \qquad \langle w_1, w_1 \rangle = 1 \qquad \langle v_2, w_1 \rangle = 1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - w_1 = (-1, 1, 0)$$

$\langle V_3, W_1\rangle = 0 \quad \langle V_3, W_2\rangle = 1 \quad \langle W_2, W_2\rangle = 0$

$$W_3 = V_3 - \frac{\langle V_3, W_1\rangle}{\langle W_1, W_1\rangle} W_1 - \frac{\langle V_3, W_2\rangle}{\langle W_2, W_2\rangle} W_2 \rightsquigarrow \begin{cases} \text{NON ESISTE PERCHÉ} \\ \mathcal{B} \text{ È INDEFINITA} \end{cases}$$

(b) $y + z = 0 \quad$ BASE: $\{ \overset{V_1}{(1,0,0)}, \overset{V_2}{(0,1,-1)} \}$

$$V_3 = (a, b, c) \perp V_1, V_2 \quad \begin{cases} V_3^T \mathcal{B} V_1 = 0 \\ V_3^T \mathcal{B} V_2 = 0 \end{cases} \begin{cases} a + b = 0 \\ a - 2c = 0 \end{cases} \begin{cases} b = -2t \\ c = t \quad a = 2t \end{cases}$$

$$\rightsquigarrow V_3 = (2, -2, 1) \quad \text{IN BASE } \{V_1, V_2, V_3\} \quad P_\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{IN BASE CANONICA: } M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix} \qquad M^{-1} = -1 \begin{pmatrix} -1 & -0 & 0 \\ -2 & 1 & +1 \\ -2 & +2 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & -1 & -1 \end{pmatrix}$$

$\det(M) = 1 - 2 = -1$

$$P = M P_\varepsilon M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix}$$

2) $B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad DET(B) = 3 - 1 - 1 = 1$

(a) SYLVESTER(1,2,3) : $DET(1)=1 \quad DET\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} = 2 \quad DET(B) = 1$

$\overset{}{+}\overset{P}{+}\overset{P}{+}\overset{P}{+} \rightsquigarrow m_+ = 3 \quad m_- = 0 \quad m_0 = 0 \rightsquigarrow$ DEF. POSITIVA

(b)

$$B_\varepsilon = \overset{M^T}{\begin{pmatrix} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}} \overset{B}{\begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}} \overset{M}{\begin{pmatrix} -1 & 3 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}} =$$

$$= \begin{pmatrix} -3 & 7 & 2 \\ 3 & -5 & -2 \\ 0 & 3 & 2 \end{pmatrix} \begin{pmatrix} -1 & 3 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 17 & -13 & 6 \\ -13 & 13 & -5 \\ 6 & -5 & 5 \end{pmatrix}$$

(c)

$$x^T B v = (x\ y\ z) \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = (x\ y\ z) \begin{pmatrix} -2 \\ 7 \\ 5 \end{pmatrix} = 0$$

$\rightsquigarrow -2x + 7y + 5z = 0$

(d) $x = 3y - z \quad$ BASE: $v_1 = (1, 0, -1) \quad v_2 = (3, 1, 0)$

$$\langle v_1, v_1 \rangle = v_1^T B v_1 = (1\ 0\ -1) \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = 2 \qquad \langle v_1, v_2 \rangle = v_2^T B v_1 = 1$$

$$w_1 = v_1 \qquad \hat{w}_2 = v_2 - \frac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1 = (3,1,0) - \frac{1}{2}(1,0,-1) = \left(\frac{5}{2}, 1, \frac{1}{2}\right)$$

BASE ORTOGONALE : $w_1 = (1, 0, -1) \quad w_2 = (5, 2, 1)$

(e) $v_1 = (1, 0, 0) \quad v_2 = (0, 1, 1)$

$$x^T B v_1 = (x\ y\ z) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = x - y = 0 \qquad x^T B v_2 = (x\ y\ z) \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} = -x + 5y + 2z = 0$$

$$\begin{cases} x - y = 0 \\ -x + 5y + 2z = 0 \end{cases} \rightsquigarrow \begin{cases} x = \delta \quad y = \delta \\ z = -\frac{3}{2}\delta \end{cases} \rightsquigarrow (2, 2, -3)$$

(f) $w = (1,0,0) = w_\perp + w_{/\!/} \quad w_{/\!/} = (0,a,b) \in SPAN\{\overset{v_1}{(0,1,0)}, \overset{v_2}{(0,0,1)}\}$

$w_\perp = w - w_{/\!/} = (1,-a,-b)$

$$\begin{cases} w_\perp^T B v_1 = 0 \\ w_\perp^T B v_2 = 0 \end{cases} \quad \begin{cases} -1-3a-b=0 \\ -a-b=0 \end{cases} \quad \begin{cases} -1-2a=0 & a=-1/2 \\ b = 1/2 \end{cases}$$

$\rightsquigarrow \ w_{/\!/} = (0, -1/2, 1/2)$

(g) $v_1 = (1,0,0) \quad v_2 = (0,1,0) \quad v_3 = (0,0,1)$

$w_1 = v_1 = (1,0,0) \qquad \langle w_1, w_1\rangle = 1 \qquad \langle v_2, w_1\rangle = -1$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = v_2 + w_1 = (1,1,0)$$

$$\langle v_3, w_1\rangle = 0 \quad \langle v_3, w_2\rangle = 1 \quad \langle w_2, w_2\rangle = (1,1,0)\begin{pmatrix}0\\2\\1\end{pmatrix} = 2$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 = v_3 - \frac{1}{2} w_2 = (-1/2, -1/2, 1)$$

$$\langle w_3, w_3\rangle = (-1/2, -1/2, 1)\begin{pmatrix}0\\0\\1/2\end{pmatrix} = 1/2$$

$$w_1^* = \frac{w_1}{\|w_1\|} = (1,0,0) \quad w_2^* = \frac{w_2}{\|w_2\|} = \left(\frac{\sqrt2}{2}, \frac{\sqrt2}{2}, 0\right) \quad w_3^* = \frac{w_3}{\|w_3\|} = \left(-\frac{\sqrt2}{2}, -\frac{\sqrt2}{2}, \sqrt2\right)$$

(h) $y + z = 0 \qquad BASE: \{\overset{v_1}{(1,0,0)}, \overset{v_2}{(0,1,-1)}\}$

$$v_3 = (a,b,c) \perp v_1, v_2 \quad \begin{cases} v_3^T B v_1 = 0 \\ v_3^T B v_2 = 0 \end{cases} \quad \begin{cases} a - b = 0 \\ -a + 2b = 0 \end{cases} \quad a = b = 0$$

$$\rightsquigarrow v_3 = (0,0,1) \quad IN\ BASE\ \{v_1, v_2, v_3\} \quad P_E = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix}$$

$$IN\ BASE\ CANONICA: \ M = \begin{pmatrix} 1&0&0\\0&1&0\\0&-1&1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1&-0&0\\-0&1&-1\\0&-0&1 \end{pmatrix}^T = \begin{pmatrix} 1&0&0\\0&1&0\\0&-1&1 \end{pmatrix}$$

$$P = M P_E M^{-1} = \begin{pmatrix} 1&0&0\\0&1&0\\0&-1&0 \end{pmatrix}\begin{pmatrix} 1&0&0\\0&1&0\\0&-1&1 \end{pmatrix} = \begin{pmatrix} 1&0&0\\0&1&0\\0&-1&0 \end{pmatrix}$$

3) $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ $\quad DET(B) = 6+1+1-2-1-3 = 2$

(a) SYLVESTER(1,2,3) : $DET(1)=1 \quad DET\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 1 \quad DET(B) = 2$

$\overset{\quad p \ \ p \ \ p}{+ + + +} \leadsto n_+ = 3 \quad n_- = 0 \quad n_0 = 0 \leadsto$ DEF. POSITIVA

(b) $B_\varepsilon = \overset{M^T}{\begin{pmatrix} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}} \overset{B}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}} \overset{M}{\begin{pmatrix} -1 & 3 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}} =$

$= \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & -3 \\ 3 & 5 & 5 \end{pmatrix} \begin{pmatrix} -1 & 3 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 5 \\ 1 & 9 & -1 \\ 5 & -1 & 12 \end{pmatrix}$

(c)

$x^T B v = (x \ y \ z) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = (x \ y \ z) \begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix} = 0$

$\leadsto 3x + 5y + 9z = 0$

(d) $x = 3y - z$ BASE: $v_1 = (1, 0, -1) \quad v_2 = (3, 1, 0)$

$\langle v_1, v_1 \rangle = v_1^T B v_1 = (1 \ 0 \ -1) \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = 2 \quad \langle v_1, v_2 \rangle = v_2^T B v_1 = 0$

$w_1 = v_1 \qquad \hat{w}_2 = v_2 - \frac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1 = (3, 1, 0)$

BASE ORTOGONALE : $w_1 = (1, 0, -1) \quad w_2 = (3, 1, 0)$

(e) $v_1 = (1, 0, 0) \quad v_2 = (0, 1, 1)$

$x^T B v_1 = (x \ y \ z) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x + y + z = 0 \quad x^T B v_2 = (x \ y \ z) \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = 2x + 3y + 5z = 0$

$\begin{cases} x + y + z = 0 \\ 2x + 3y + 5z = 0 \end{cases} \leadsto \begin{cases} y + 2z = 0 \\ 2x - 2z = 0 \end{cases} \leadsto (1, -2, 1)$

(f) $w=(1,0,0)=w_\perp+w_{\parallel}$ $\quad w_{\parallel}=(0,a,b)\in SPAN\{\overset{v_1}{(0,1,0)},\overset{v_2}{(0,0,1)}\}$

$w_\perp = w-w_{\parallel}=(1,-a,-b)$

$$\begin{cases} w_\perp^T B v_1 = 0 \\ w_\perp^T B v_2 = 0 \end{cases} \quad \begin{cases} 1-2a-b=0 \\ 1-a-3b=0 \end{cases} \quad \begin{cases} 2a=4/5 \quad a=2/5 \\ 1-5b=0 \quad b=1/5 \end{cases}$$

$\rightsquigarrow w_{\parallel}=(0,2/5,1/5)$

(g) $v_1=(1,0,0)$ $\quad v_2=(0,1,0)$ $\quad v_3=(0,0,1)$

$w_1=v_1=(1,0,0)$ $\quad \langle w_1,w_1\rangle=1$ $\quad \langle v_2,w_1\rangle=1$

$$w_2=v_2-\frac{\langle v_2,w_1\rangle}{\langle w_1,w_1\rangle}w_1=v_2-w_1=(-1,1,0)$$

$$\langle v_3,w_1\rangle=1 \quad \langle v_3,w_2\rangle=0 \quad \langle w_2,w_2\rangle=(-1,1,0)\begin{pmatrix}0\\1\\0\end{pmatrix}=1$$

$$w_3=v_3-\frac{\langle v_3,w_1\rangle}{\langle w_1,w_1\rangle}w_1-\frac{\langle v_3,w_2\rangle}{\langle w_2,w_2\rangle}w_2=v_3-w_1=(-1,0,1)$$

$$\langle w_3,w_3\rangle=(-1,0,1)\begin{pmatrix}0\\0\\2\end{pmatrix}=2$$

$$w_1^*=\frac{w_1}{\|w_1\|}=(1,0,0) \quad w_2^*=\frac{w_2}{\|w_2\|}=(-1,1,0) \quad w_3^*=\frac{w_3}{\|w_3\|}=\left(-\frac{\sqrt2}{2},0,\frac{\sqrt2}{2}\right)$$

(h) $y+z=0$ $\quad$ BASE: $\{\overset{v_1}{(1,0,0)},\overset{v_2}{(0,1,-1)}\}$

$$v_3=(a,b,c)\perp v_1,v_2 \quad \begin{cases} v_3^T B v_1 = 0 \\ v_3^T B v_2 = 0 \end{cases} \quad \begin{cases} a+b+c=0 \\ b-2c=0 \end{cases} \quad \begin{cases} a=-3\delta \\ c=\delta \quad b=2\delta \end{cases}$$

$$\rightsquigarrow v_3=(-3,2,1) \quad \text{IN BASE } \{v_1,v_2,v_3\} \quad P_\varepsilon=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

IN BASE CANONICA: $M=\begin{pmatrix}1&0&-3\\0&1&2\\0&-1&1\end{pmatrix}$ $\quad M^{-1}=\frac{1}{3}\begin{pmatrix}3&-0&0\\+3&1&+1\\3&-2&1\end{pmatrix}^T=\frac{1}{3}\begin{pmatrix}3&3&3\\0&1&-2\\0&1&1\end{pmatrix}$

$DET(M)=1+2=3$

$$P=MP_\varepsilon M^{-1}=\frac{1}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&-1&0\end{pmatrix}\begin{pmatrix}3&3&3\\0&1&-2\\0&1&1\end{pmatrix}=\begin{pmatrix}1&1&1\\0&1/3&-2/3\\0&-1/3&2/3\end{pmatrix}$$

5) $B = \begin{pmatrix} 6 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}$ $\quad DET(B) = 30 - 25 - 5 = 1$

(a) SYLVESTER$(1,2,3)$: $DET(6)=6$ $\quad DET\begin{pmatrix} 6 & 1 \\ 1 & 1 \end{pmatrix} = 5$ $\quad DET(B)=1$

$\overset{}{+}\overset{P}{+}\overset{P}{+}\overset{P}{+} \rightsquigarrow n_+ = 3 \quad n_- = 0 \quad n_0 = 0 \rightsquigarrow$ DEF. POSITIVA

(b)

$$B_{\mathcal{E}} = \overset{M^T}{\begin{pmatrix} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}} \overset{B}{\begin{pmatrix} 6 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}} \overset{M}{\begin{pmatrix} -1 & 3 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}} =$$

$$= \begin{pmatrix} -5 & 1 & 5 \\ 18 & -1 & -10 \\ 7 & 5 & 7 \end{pmatrix} \begin{pmatrix} -1 & 3 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -20 & 1 \\ -20 & 75 & 7 \\ 1 & 7 & 18 \end{pmatrix}$$

(c)

$$x^T B v = (x\ y\ z) \begin{pmatrix} 6 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = (x\ y\ z) \begin{pmatrix} -5 \\ 6 \\ 17 \end{pmatrix} = 0$$

$\rightsquigarrow -5x + 6y + 17z = 0$

(d) $x = 3y - z$ $\quad$ BASE: $v_1 = (1, 0, -1)$ $\quad v_2 = (3, 1, 0)$

$$\langle v_1, v_1 \rangle = v_1^T B v_1 = (1\ 0\ -1) \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} = 11 \quad \langle v_1, v_2 \rangle = v_2^T B v_1 = 17$$

$$w_1 = v_1 \qquad \hat{w}_2 = v_2 - \frac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1 = v_2 - \frac{17}{11} v_1 = \left(\frac{16}{11}, 1, \frac{17}{11} \right)$$

BASE ORTOGONALE: $w_1 = (1, 0, -1)$ $\quad w_2 = (16, 11, 17)$

(e) $v_1 = (1, 0, 0)$ $\quad v_2 = (0, 1, 1)$

$$x^T B v_1 = (x\ y\ z) \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} = 6x + y = 0 \qquad x^T B v_2 = (x\ y\ z) \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = x + y + 2z = 0$$

$$\begin{cases} 6x + y = 0 \\ x + y + 2z = 0 \end{cases} \rightsquigarrow \begin{cases} y = -6x \\ z = \frac{5}{2} x \end{cases} \rightsquigarrow (2, -12, 5)$$

(f) $w = (1,0,0) = w_\perp + w_{//} \qquad w_{//} = (0,a,b) \in \text{SPAN}\{\overset{v_1}{(0,1,0)}, \overset{v_2}{(0,0,1)}\}$

$w_\perp = w - w_{//} = (1,-a,-b)$

$$\begin{cases} w_\perp^\delta B v_1 = 0 \\ w_\perp^\delta B v_2 = 0 \end{cases} \quad \begin{cases} 1-a-2b=0 \\ -2a-5b=0 \end{cases} \quad \begin{cases} 2+b=0 \quad b=-2 \\ a=5 \end{cases}$$

$\leadsto \; w_{//} = (0,5,-2)$

(g) $v_1 = (1,0,0) \quad v_2 = (0,1,0) \quad v_3 = (0,0,1)$

$w_1 = v_1 = (1,0,0) \qquad \langle w_1, w_1 \rangle = 6 \qquad \langle v_2, w_1 \rangle = 1$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - \frac{1}{6} w_1 = (-1/6, 1, 0)$$

$\langle v_3, w_1 \rangle = 0 \qquad \langle v_3, w_2 \rangle = 2 \qquad \langle w_2, w_2 \rangle = (-1/6, 1, 0)\begin{pmatrix} 0 \\ 5/6 \\ 2 \end{pmatrix} = 5/6$

$$\hat{w}_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = v_3 - \frac{12}{5} w_2 = \left(\frac{2}{5}, -\frac{12}{5}, 1\right)$$

$w_3 = (2,-12,5) \qquad \langle w_3, w_3 \rangle = (2,-12,5)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 5$

$$w_1^* = \frac{w_1}{\|w_1\|} = \left(\frac{1}{\sqrt{6}}, 0, 0\right) \quad w_2^* = \frac{w_2}{\|w_2\|} = \left(-\frac{\sqrt{6}}{6\sqrt{5}}, \frac{\sqrt{6}}{\sqrt{5}}, 0\right) \quad w_3^* = \frac{w_3}{\|w_3\|} = \left(\frac{2}{\sqrt{5}}, \frac{-12}{\sqrt{5}}, \frac{5}{\sqrt{5}}\right)$$

(h) $y + z = 0 \qquad \text{BASE: } \{\overset{v_1}{(1,0,0)}, \overset{v_2}{(0,1,-1)}\}$

$$v_3 = (a,b,c) \perp v_1, v_2 \qquad \begin{cases} v_3^\delta B v_1 = 0 \\ v_3^\delta B v_2 = 0 \end{cases} \quad \begin{cases} 6a+b=0 \\ a+b+2c=0 \end{cases} \quad \begin{cases} 5a = 2c \quad a = \frac{2c}{5} \\ b = -\frac{12}{5}c \end{cases}$$

$$\leadsto v_3 = (2,-12,5) \quad \text{IN BASE } \{v_1, v_2, v_3\} \quad P_\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

IN BASE CANONICA: $M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -12 \\ 0 & -1 & 5 \end{pmatrix} \quad M^{-1} = -\frac{1}{7}\begin{pmatrix} -7 & -0 & 0 \\ -2 & 5 & +1 \\ -2 & +12 & 1 \end{pmatrix}^\delta = \frac{1}{7}\begin{pmatrix} 7 & 2 & 2 \\ 0 & -5 & -12 \\ 0 & -1 & -1 \end{pmatrix}$

$\text{DET}(M) = 5 - 12 = -7$

$$P = M P_\varepsilon M^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 7 & 2 & 2 \\ 0 & -5 & -12 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2/7 & 2/7 \\ 0 & -5/7 & -12/7 \\ 0 & 5/7 & 12/7 \end{pmatrix}$$