Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 20 September 2022

1. Let us consider the functional

$$F(u) = \int_0^1 \left(u'(x)^2 + x^2 u(x)\right)\,dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) + u(1) = 2$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) - u(1) = 2$.

2. Discuss existence, uniqueness, and regularity of solutions to the boundary value problem

$$(1 + u'(x)^2) \cdot u''(x) = u(x)^3 + \sin^6 x, \qquad u'(0) = u(\pi) = 6.$$

3. Let B be an open ball in $\mathbb{R}^2$. For every real number $p \geq 1$, let us set

$$I(p) := \inf\left\{\int_B \left(|\nabla u|^p + u^2\right)\,dx\,dy : u \in W_0^{1,p}(B) \cap L^{20}(B),\ \int_B u^{20}\,dx\,dy \geq 20\right\}.$$

   (a) Determine whether there exists p such that $I(p) > 0$.

   (b) Determine whether there exists p such that $I(p) = 0$.

   (c) Determine for which values of p it turns out that $I(p)$ is actually a minimum.

4. For every function $f : [0,1] \to \mathbb{R}$, let us set

$$[Tf](x) := \int_0^x \cos t \cdot f(t)\,dt \qquad \forall x \in [0,1].$$

   Determine whether the restriction of T defines

   (a) a strong-strong continuous operator $L^2((0,1)) \to L^\infty((0,1))$ (and in case compute its norm),

   (b) a weak-strong continuous operator $L^{37}((0,1)) \to L^{73}((0,1))$,

   (c) a compact operator $L^5((0,1)) \to L^{50}((0,1))$,

   (d) a compact operator $C^0([0,1]) \to H^1((0,1))$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.