Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 20 September 2022

1. Let us consider the functional

$$F(u) = \int_0^1 \left(u'(x)^2 + x^2 u(x)\right) dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) + u(1) = 2$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) - u(1) = 2$.

2. Discuss existence, uniqueness, and regularity of solutions to the boundary value problem

$$(1 + u'(x)^2) \cdot u''(x) = u(x)^3 + \sin^6 x, \qquad u'(0) = u(\pi) = 6.$$

3. Let B be an open ball in $\mathbb{R}^2$. For every real number $p \geq 1$, let us set

$$I(p) := \inf \left\{ \int_B \left(|\nabla u|^p + u^2\right) dx\, dy : u \in W_0^{1,p}(B) \cap L^{20}(B),\ \int_B u^{20}\, dx\, dt \geq 20 \right\}.$$

   (a) Determine whether there exists p such that $I(p) > 0$.

   (b) Determine whether there exists p such that $I(p) = 0$.

   (c) Determine for which values of p it turns out that $I(p)$ is actually a minimum.

4. For every function $f : [0, 1] \to \mathbb{R}$, let us set

$$[Tf](x) := \int_0^x \cos t \cdot f(t)\, dt \qquad \forall x \in [0, 1].$$

   Determine whether the restriction of T defines

   (a) a strong-strong continuous operator $L^2((0, 1)) \to L^\infty((0, 1))$ (and in case compute its norm),

   (b) a weak-strong continuous operator $L^{37}((0, 1)) \to L^{73}((0, 1))$,

   (c) a compact operator $L^5((0, 1)) \to L^{50}((0, 1))$,

   (d) a compact operator $C^0([0, 1]) \to H^1((0, 1))$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.

1. Let us consider the functional

$$F(u) = \int_0^1 \left(u'(x)^2 + x^2 u(x)\right)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) + u(1) = 2$.

(b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) - u(1) = 2$.

(a) $\delta F(u, v) = \int_0^1 2\dot{u}\dot{v} + x^2 v = \int_0^1 (-2\ddot{u} + x^2)v + 2(\dot{u}(1)v(1) - \dot{u}(0)v(0))$

for every $v \in C^1([0,1])$ with $v(0) + v(1) = 0$, namely $v(1) = -v(0)$

In the usual way we obtain

$$\begin{cases} \ddot{u} = \frac{1}{2}x^2 & \leftarrow \text{ELE} \\ u(0) + u(1) = 2 & \leftarrow \text{given BC} \\ \dot{u}(1) = -\dot{u}(0) & \leftarrow \text{BC "on the road"} \end{cases}$$

$\rightsquigarrow u(x) = \frac{1}{24}x^4 + a + bx$

$\dot{u}(x) = \frac{1}{6}x^3 + b$

Imposing BCs we obtain

$\frac{1}{6} + b = -b \qquad a + \frac{1}{24} + a + b = 2 \rightsquigarrow$ find a and b in a unique way

The corresponding $u_0(x) = \frac{1}{24}x^4 + a + bx$ is the unique min. point because

$$F(\underbrace{u_0 + v}_{\text{any competitor}}) = F(u_0) + \underbrace{\delta F(u_0, v)}_{=0} + \int_0^1 \dot{v}^2 \geq F(u_0)$$

with equality iff $v \equiv$ constant, but the only constant with $v(0) + v(1) = 0$ is 0.

(b) $\inf = -\infty$ and a possible minimizing sequence is

$u_n(x) = -n - 2x$

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2. Discuss existence, uniqueness, and regularity of solutions to the boundary value problem

$$(1+u'(x)^2)\cdot u''(x) = u(x)^3 + \sin^6 x, \qquad u'(0) = u(\pi) = 6.$$

$(L_p)' = L_s$ ELE and $L_p = 0$ NBC

$$L(x,s,p) = \frac{1}{2}p^2 + \frac{1}{12}p^4 - 78p + \frac{1}{4}s^4 + \sin^6 x\cdot s$$

$(L_p)' = \left(\dot u + \frac{\dot u^3}{3} - 78\right)' = \ddot u + \dot u^2 \ddot u$ $\qquad L_p = 0 \iff \dot u + \frac{\dot u^3}{3} = 78$

$\iff \dot u = 6$

Variational formulation

$$\min\left\{ \int_0^\pi \left[\frac{\dot u^2}{2} + \frac{\dot u^4}{12} - 78\dot u + \frac{u^4}{4} + \sin^6 x\cdot u\right]dx : u \in H^1((0,\pi)),\ \underbrace{u(\pi)=6}\right\}$$

↑ well-defined in $H^1((0,\pi))$

We allow $+\infty$ as a possible value of the functional

$\boxed{\text{Compactness}}$ $L(x,s,p) \geq \frac{1}{2}p^2 + \frac{1}{8}s^4 - A$

↑ suitable constant

Every sequence with $F(u_n) \leq M$ satisfies $\|\dot u_n\|_{L^2} \leq M'$ and $\|u_n\|_{L^4} \leq M''$

In the usual way this implies that

$u_{n_k} \to u_\infty$ unif in $[0,\pi]$ $\qquad \dot u_{n_k} \rightharpoonup \dot u_\infty$ weak L^2

The BC passes to the limit

$\boxed{\text{LSC}}$ Rather standard (convexity wrt p and unif. conv. in u)

This implies the existence of at least one minimizer

$\boxed{\text{Regularity}}$ Standard procedure. Computing $F(u_0+tv)$ and letting $t\to 0$, with $v \in C^\infty_c((0,\pi))$ we find

$\left(\dot u + \frac{\dot u^3}{3} - 78\right)' = u^3 + \sin^6 x$ ⤳ ELE in weak form

$u \in H^1 \Rightarrow \text{RHS} \in C^0 \Rightarrow \dot u + \frac{\dot u^3}{3} - 78 \in C^1 \Rightarrow \dot u \in C^1 \Rightarrow u \in C^2$

↑ because $\varphi(p) = p + \frac{1}{3}p^3$ is C^∞ with inverse of class C^∞ ($\varphi'(p) \geq 1$ for every $p \in \mathbb{R}$)

$\Rightarrow$ (bootstrap) $u \in C^\infty$

In a similar way we obtain the NBC

$\boxed{\text{Uniqueness}}$ Every solution is a minimizer, which are unique because

$$F(u_0+v) \geq F(u_0) + \delta F(u_0, v) + \int_0^\pi \dot v^2 \geq F(u_0)$$

with equality $\iff \dot v \equiv 0 \iff v \equiv \text{constant} \iff v \equiv 0$ (because $v(\pi) = 0$)

3. Let B be an open ball in $\mathbb{R}^2$. For every real number $p \geq 1$, let us set

$$I(p) := \inf \left\{ \int_B \left(|\nabla u|^p + u^2 \right) dx\, dy : u \in W_0^{1,p}(B) \cap L^{20}(B),\ \int_B u^{20}\, dx\, dt \geq 20 \right\}.$$

(a) Determine whether there exists p such that $I(p) > 0$.

(b) Determine whether there exists p such that $I(p) = 0$.

(c) Determine for which values of p it turns out that $I(p)$ is actually a minimum.

(a) $\boxed{\text{YES, every } p \geq \frac{20}{11}}$ namely whenever $p_* \geq 20$. Indeed

$$\int_B |\nabla u|^p + u^{20} \geq \int_B |\nabla u|^p = \|\nabla u\|_{L^p}^p \geq C \|u\|_{L^{20}}^p \geq C \cdot 20^{p/20}$$

Poincaré – Sobolev

(b) $\boxed{\text{YES, every } 1 \leq p < \frac{20}{11}}$ Indeed, consider any nontrivial $\varphi \in C_c^\infty(B)$. Assume wlog that B has center in the origin and that $\int_B \varphi^{20} \geq 20$ (otherwise consider $M\varphi$, with M large). Set $u_\lambda(x) := \lambda^a \varphi(\lambda x)$ with $\lambda \geq 1$. Then $u_\lambda \in W_0^{1,p}(B)$ and

$$\int_B u_\lambda^{20} = \lambda^{20a-2} \int_B \varphi^{20} \ ; \quad \int_B u_\lambda^2 = \lambda^{2a-2} \int_B \varphi^2 ; \quad \int_B |\nabla u_\lambda|^p = \lambda^{(a+1)p-2} \int_B |\nabla \varphi|^p$$

Setting $a = \frac{1}{10}$, when $p < \frac{20}{11}$ we obtain that $\int_B u_\lambda^{20} \geq 20$ and

$$\int_B |\nabla u_\lambda|^p + u_\lambda^2 \to 0,$$ as required

(c) $\boxed{\text{Inf} = \min \iff p > \frac{20}{11}}$ We distinguish 3 cases.

- If $p < \frac{20}{11}$, from point (b) we know that $\inf = 0$, which can not be a minimum (u would be a constant)
- If $p > \frac{20}{11}$, then the embedding $W_0^{1,p} \to L^{20}$ is compact. In this case the standard direct method works, because the condition $\int u^{20} \geq 20$ passes to the limit.
- If $p = \frac{20}{11}$, then the same argument of point (b) shows that, if $u(x)$ is a competitor, then $\lambda^{1/10} u(\lambda x)$ is a better competitor for every $\lambda > 1$ (because $\int u^2$ decreases).

4. For every function $f : [0,1] \to \mathbb{R}$, let us set

$$[Tf](x) := \int_0^x \cos t \cdot f(t)\, dt \qquad \forall x \in [0,1].$$

Determine whether the restriction of T defines

(a) a strong-strong continuous operator $L^2((0,1)) \to L^\infty((0,1))$ (and in case compute its norm),

(b) a weak-strong continuous operator $L^{37}((0,1)) \to L^{73}((0,1))$,

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(d) a compact operator $C^0([0,1]) \to H^1((0,1))$.

(a) $\boxed{\text{YES}}$ $\displaystyle |[Tf](x)| \le \int_0^x |\cos t| \cdot |f(t)|\, dt \le \int_0^1 |\cos t| \cdot |f(t)|\, dt$

$$\le \boxed{\left\{ \int_0^1 \cos^2 t \, dt \right\}^{1/2}} \cdot \left\{ \int_0^1 f(t)^2 dt \right\}^{1/2} \qquad \forall x \in (0,1)$$

↑ norm of the operator because we have equality when $f(x) = \cos x$ and $x = 1$

(b) $\boxed{\text{YES}}$ If $f_n \to f_\infty$ weakly in L^{37}, then $\|f_n\|_{L^{37}}$ is bounded and therefore $\|Tf_n\|_{L^\infty}$ is bounded (Hölder inequality)

Moreover

$$[Tf_n](x) \to \int_0^x \cos t \cdot f_\infty(t)\, dt = [Tf_\infty](x) \qquad \forall x \in (0,1)$$

Therefore $Tf_n \to Tf_\infty$ in the pointwise sense with L^∞ domination, and hence $Tf_n \to Tf_\infty$ in L^{73}.

(c) $\boxed{\text{YES}}$ If $\{f_n\} \subseteq L^5$ is bounded, then $\{Tf_n\}$ is equi-bounded and equi-continuous. Conclusion follows from Ascoli-Arzelà.

$$|Tf_n(x) - Tf_n(y)| \le \int_x^y |\cos t| \cdot |f_n(t)|\, dt \le \left\{ \int_x^y |\cos t|^{5/4} \right\}^{\frac{4}{5}} \|f_n\|_{L^5}$$

$$\le |y-x|^{4/5}$$

(d) $\boxed{\text{NO}}$ The sequence $f_n(x) = \sin(nx)$ is bounded in $C^0([0,1])$.

Now observe that $[Tf_n]'(x) = \cos t \cdot \sin(nx) \rightharpoonup 0$.

As a consequence, the only possible limit in L^2 of a subsequence $[Tf_{n_k}]'$ is 0, but its norm does not tend to 0

$$\|Tf_n\|_{L^2}^2 \ge \int_0^{\frac{\pi}{4}} \cos^2 t \sin^2(nt)\, dt \ge \frac{1}{2} \int_0^{\frac{\pi}{4}} \sin^2(nt)\, dt \sim \frac{\pi}{16}$$

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