1. Let us consider the functional

$$F(u) = \int_0^{\pi} \left(u'(x)^2 + u(x)^2 - \sin x \cdot u(x)\right)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) - 2u(\pi) = 3$.

(b) Discuss the minimum problem for $F(u)$ with boundary condition $u'(0) - 2u'(\pi) = 3$.

(a) $$\delta F(u,v) = \int_0^{\pi} (2\dot{u}\dot{v} + 2uv - \sin x \cdot v)$$

$$= \int_0^{\pi} (-2\ddot{u} + 2u - \sin x)\, v + 2\left(\dot{u}(\pi)v(\pi) - \dot{u}(0)\underset{2v(\pi)}{\underbrace{v(0)}}\right)$$

$\rightsquigarrow$ every min. point solves

$$\begin{cases} \ddot{u} = u - \frac{1}{2}\sin x & \text{(ELE)} \\ u(0) - 2u(\pi) = 3 & \text{(given BC)} \\ \dot{u}(\pi) - 2\dot{u}(0) = 0 & \text{(BC originated on the road)} \end{cases}$$

$\rightsquigarrow$ $u(x) = a\cosh x + b\sinh x + \frac{1}{4}\sin x$

BC $\rightsquigarrow$ unique values of a, b (or coerciveness $\rightsquigarrow$ at least one solution)

The solution is the unique min. point because

$$F(u+v) = F(u) + \delta F(u,v) + \int_0^{\pi} (\dot{v}^2 + v^2)$$

(b) The min does not exist, and the inf. is equal to the min. of $F(u)$ without BCs, which in turn is realized by the unique solution to

$$\ddot{u} = u - \frac{1}{2}\sin x$$

$\dot{u}(0) = 0$

$\dot{u}(\pi) = 0$

(NBC) on the road } incompatible with given BC

A possible way to find a minimizing sequence consists in taking a min. u without BC, and then modify it near $x = 0$ so that $\dot{u}(0) = 3$

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2. Discuss existence, uniqueness, and regularity of solutions to the boundary value problem

$$u'' = u^3 + \sin^6 x, \qquad u'(3) = 6, \quad u'(6) = 3.$$

Let us consider the variational pbm

$$\min \left\{ \int_3^6 \left(\tfrac{1}{2}\dot{u}^2 + (x-9)\dot{u} + \tfrac{1}{4}u^4 + \sin^6 x\, u + u \right) : u \in H^1((3,6)) \right\}$$

namely with Lagrangian $L(x,s,p) := \frac{1}{2}p^2 + (x-9)p + \frac{s^4}{4} + (\sin^6 x + 1)s$

Then (ELE) is the given equation, and NBC are $\dot{u} = 9 - x$ for $x = 3$ and $x = 6$, namely the given BC.

Existence of minimizers, uniqueness and regularity are rather standard.
In particular, compactness wrt the usual notion of convergence follows from the estimate

$$L(x,s,p) \geq \frac{1}{2}p^2 - 6|p| + \frac{s^4}{4} - 2|s|$$

$$\geq \frac{1}{4}p^2 + \frac{1}{8}s^4 - A.$$

Alternative approach Set $u(x) = v(x) + \boxed{9x - \frac{x^2}{2}}$
and observe that satisfies the NBC

$$\ddot{v} = \ddot{u} + 1 = u^3 + \sin^6 x + 1 = \left(v + 9x - \frac{x^2}{2}\right)^3 + \sin^6 x + 1,$$

which leads to the variational formulation

$$\min \left\{ \int_3^6 \left(\tfrac{1}{2}\dot{v}^2 + \tfrac{1}{4}\left(v + 9x - \tfrac{x^2}{2}\right)^4 + (\sin^6 x + 1)v \right) : v \in H^1((3,6)) \right\}$$

Again it is true that $L(x,s,p) \geq \frac{1}{2}p^2 + \frac{1}{8}s^2 - A$ ...

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3. Let us consider the square $Q := (0,1)^2$. For every real number $p \geq 1$, let us consider the set

$$\mathcal{S}(p) := \left\{ u \in C_c^\infty(Q) : \int_Q \left((1+u^2)|u_x|^p + \cos y \cdot |u_y|^p \right) dx\, dy \leq 2022 \right\}.$$

(a) Determine whether there exists p such that $\mathcal{S}(p)$ is relatively compact in $L^8(Q)$.

(b) Determine whether there exists p such that $\mathcal{S}(p)$ is not relatively compact in $L^8(Q)$.

(c) Determine the weak closure of $\mathcal{S}(12)$ in $L^8(Q)$.

(a) $\boxed{\text{YES}}$ $S(p)$ is relatively cpt. in $L^8(Q)$ when $\boxed{p > \frac{8}{5}}$, namely when the embedding $W^{1,p}(Q) \to L^8(Q)$ is compact.

Indeed

$$\|u\|^p_{1,p,Q} \underset{\text{Poincaré}}{\leq} \text{cost} \|\nabla u\|^p_{L^p(Q)} \underset{\substack{\cos y \geq \cos 1 \\ \text{in } Q}}{\leq} \text{cost} \int_Q \left((1+u^2)|u_x|^p + \cos y |u_y|^p\right) \underset{\text{defn. of } S(p)}{\leq} \text{cost}$$

Therefore, $S(p)$ is bounded in $W^{1,p}(Q)$, and hence it is relatively cpt. in $L^8(Q)$ whenever $8 < p^*$.

(b) $\boxed{\text{YES}}$ $S(p)$ is not relatively cpt. in $L^8(Q)$ if $\boxed{p < \frac{6}{5}}$

Let $\varphi \in C_c^\infty(Q)$ with $\varphi \not\equiv 0$. Let $u(x,y) = \lambda^a \varphi(\lambda x, \lambda y)$, which is well defined and in $C_c^\infty(Q)$ for every $\lambda \geq 1$.

With a standard change of variables we obtain that

$$\int_Q u^8 = \lambda^{8a-2} \int_Q \varphi^8 \qquad \int_Q u^2 |u_x|^p = \lambda^{2a+(a+1)p-2} \int_Q \varphi^2 |\varphi_x|^p$$

$$\int_Q |u_x|^p = \lambda^{(a+1)p-2} \int_Q |\varphi_x|^p \qquad \int_Q |u_y|^p = \lambda^{(a+1)p-2} \int_Q |\varphi_y|^p$$

As a consequence, if $\boxed{8a-2>0}$ and $\boxed{2a+(a+1)p-2<0}$, we obtain a family that is contained in $S(p)$ and unbounded in $L^8(Q)$.

The two inequalities can be satisfied simultaneously if $p < \frac{6}{5}$.

$\boxed{\text{Remark}}$ It could be interesting to investigate the range

$$\frac{6}{5} \leq p \leq \frac{8}{5}$$

(c) $\boxed{\text{It is the set of all functions } u \in W_0^{1,12}(Q) \text{ that satisfy the same inequality.}}$

Let $\hat{S}(12)$ denote this new set. We have to check two facts.

(i) If $\{u_n\} \subseteq \hat{S}(12)$ and $u_n \to u_\infty$ in $L^8(Q)$, then $u_\infty \in \hat{S}(12)$.

Indeed, as in point (a) the sequence $\{u_n\}$ is bounded in $W^{1,12}(Q)$, and therefore it is relatively cpt. in $L^8(Q)$, and also in $C^0(Q)$. The bound in $W^{1,12}(Q)$ implies that the limit u_∞ belongs to $W^{1,12}(Q)$. It remains to show that u_∞ satisfies the same inequality. This is true because the functional

$$u \to \int_Q (1+u^2)|u_x|^p + \cos y\, |u_y|^p$$

is LSC with respect to unif. conv. of functions and weak convergence in $L^{12}(Q)$ of gradients. The only nontrivial part is the term with $u^2|u_x|^p$, for which we can exploit the following result.

<u>Lemma</u> If $f_n \to f_\infty$ unif. and $g_n \to g_\infty$ weakly L^1, then

$$f_n g_n \to f_\infty g_\infty \text{ weakly } L^1.$$

(ii) If $u \in \hat{S}(12)$, then there exists $\{u_n\} \subseteq S(12)$ such that $u_n \to u$ weakly in L^8 (and actually also in a stronger sense, namely unif. and L^{12} weaky on the gradients).

If u satisfies the strict inequality, then it is enough to consider any approximating sequence in the definition of $W_0^{1,12}(Q)$.

Otherwise, we observe that λu with $\lambda \in (0,1)$ belongs to $\hat{S}(12)$ with strict inequality, and we conclude with the standard diagonal procedure.

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4. For every function $f : (0,1) \to \mathbb{R}$, let us set

$$[Tf](x) := \int_0^x f(\sin t)\, dt \qquad \forall x \in (0,\pi).$$

Determine whether the restriction of T defines

(a) a continuous operator $L^\infty((0,1)) \to L^2((0,\pi))$ (and in case compute its norm),
(b) a compact operator $L^\infty((0,1)) \to L^4((0,\pi))$,
(c) an open operator $L^\infty((0,1)) \to L^3((0,\pi))$,
(d) a continuous operator $L^p((0,1)) \to C^0([0,\pi])$ (the answer might depend on p).

To begin with, we observe that the operator is $\boxed{\text{LINEAR}}$.

(a) $\boxed{\text{YES}}$ Indeed

$$|[Tf](x)| \le \int_0^x |f(\sin t)|\, dt \le \|f\|_\infty \cdot x \qquad \forall x \in (0,\pi)$$

and therefore

norm of the operator

$$\|Tf\|_{L^2} \underset{\substack{\uparrow \\ \text{equality if } f \text{ is constant}}}{\le} \|f\|_\infty \left\{ \int_0^\pi x^2\, dx \right\}^{1/2} = \boxed{\left(\frac{\pi^3}{3}\right)^{1/2}} \|f\|_\infty$$

(b) $\boxed{\text{YES}}$ If $\{f_n\} \subseteq L^\infty((0,1))$ is bounded, then $\{Tf_n\}$ is equi-Lipschitz, and $[Tf_n](0) = 0$.

Ascoli-Arzelà $\Rightarrow$ $\{Tf_n\}$ is rel. cpt. in $C^0([0,\pi])$, and hence a fortiori in $L^4((0,\pi))$.

(c) $\boxed{\text{NO}}$ The operator is open if and only if it is surjective. In this case T is not surjective because all the elements of the image are Lipschitz continuous.

(d) $\boxed{\text{If and only if } p > 2}$ The key estimate is the following, where we assume that $x \in (0, \frac{\pi}{2})$

$$|[Tf](x)| \le \int_0^x |f(\sin t)|\, dt \underset{\substack{\uparrow \\ y = \sin t}}{=} \int_0^{\sin x} |f(y)| \frac{1}{\sqrt{1-y^2}}\, dy$$

$$\le \underbrace{\left\{ \int_0^{\sin x} |f(y)|^p\, dy \right\}^{1/p}}_{\le \|f\|_{L^p}} \underbrace{\left\{ \int_0^{\sin x} \frac{1}{(1-y^2)^{p'/2}}\, dy \right\}^{1/p'}}_{\text{bounded} \iff p'/2 < 1}$$

$$\iff p' < 2 \iff p > 2$$

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