Università di Pisa – Corso di Laurea in Matematica

Exam paper of “Istituzioni di Analisi Matematica”

Pisa, 22 July 2022

1. Let us consider the functional

$$F(u) = \int_0^\pi \left(u'(x)^2 + u(x)^2 - \sin x \cdot u(x)\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) - 2u(\pi) = 3$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u'(0) - 2u'(\pi) = 3$.

2. Discuss existence, uniqueness, and regularity of solutions to the boundary value problem

$$u'' = u^3 + \sin^6 x, \qquad u'(3) = 6, \quad u'(6) = 3.$$

3. Let us consider the square $Q := (0,1)^2$. For every real number $p \geq 1$, let us consider the set

$$\mathcal{S}(p) := \left\{ u \in C_c^\infty(Q) : \int_Q \left((1+u^2)|u_x|^p + \cos y \cdot |u_y|^p\right) dx\, dy \leq 2022 \right\}.$$

   (a) Determine whether there exists p such that $\mathcal{S}(p)$ is relatively compact in $L^8(Q)$.

   (b) Determine whether there exists p such that $\mathcal{S}(p)$ is not relatively compact in $L^8(Q)$.

   (c) Determine the weak closure of $\mathcal{S}(12)$ in $L^8(Q)$.

4. For every function $f : (0,1) \to \mathbb{R}$, let us set

$$[Tf](x) := \int_0^x f(\sin t)\, dt \qquad \forall x \in (0,\pi).$$

   Determine whether the restriction of T defines

   (a) a continuous operator $L^\infty((0,1)) \to L^2((0,\pi))$ (and in case compute its norm),

   (b) a compact operator $L^\infty((0,1)) \to L^4((0,\pi))$,

   (c) an open operator $L^\infty((0,1)) \to L^3((0,\pi))$,

   (d) a continuous operator $L^p((0,1)) \to C^0([0,\pi])$ (the answer might depend on p).

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.