1. Let us consider the functional

$$F(u) = \int_0^1 \big(u'(x)^2 + xu'(x) + x^2u(x)\big)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 4$.

(b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 4 + u(1)$.

$$L(x,s,p) = p^2 + xp + x^2 s$$

(ELE) $\quad (L_p)' = L_s \quad \rightsquigarrow \quad (2\dot{u} + x)' = x^2 \quad \rightsquigarrow \quad 2\ddot{u} + 1 = x^2$

(NBC) $\quad L_p = 0 \qquad 2\dot{u}(x) + x = 0$

(a) Any minimizer (if any) solves

$$\ddot{u} = \frac{1}{2}x^2 - \frac{1}{2} \quad \rightsquigarrow \quad u(x) = \frac{x^4}{24} - \frac{x^2}{4} + ax + \underset{\nwarrow 4}{b}$$

$u(0) = 4$

$$\dot{u}(x) = \frac{x^3}{6} - \frac{x}{2} + a$$

$2\dot{u}(1) + 1 = 0$

$$\rightsquigarrow \quad 2\left(\frac{1}{6} - \frac{1}{2} + a\right) = -1 \quad \rightsquigarrow \quad -\frac{2}{3} + 2a = -1 \quad \rightsquigarrow \quad a = -\frac{1}{6}$$

The unique solution is

$$\boxed{u_0(x) = \frac{x^4}{24} - \frac{x^2}{4} - \frac{1}{6}x + 4}$$

$$F(u_0 + v) = F(u_0) + \underbrace{\delta F(u_0, v)}_{=0} + \int_0^1 \dot{v}^2(x)\,dx \geq F(u_0)$$

with equality if and only if $\dot{v} \equiv 0 \rightsquigarrow v \equiv 0$ because $v(0) = 0$.

(b) $\boxed{\inf = -\infty}$ A possible minimizing sequence is

$$u_n(x) = -n - 4x.$$

2. Discuss existence, uniqueness, and regularity of functions $u : \mathbb{R} \to \mathbb{R}$ that are periodic and satisfy

$$u'' = u^3 + \sin^6 x \qquad \forall x \in \mathbb{R}.$$

Step 1 The variational pbm

$$\min\left\{ \int \left(\tfrac{1}{2}\dot{u}^2 + \tfrac{1}{4}u^4 + \sin^6 x \cdot u \right) : u \in H^1((0,\pi)),\ u(0) = u(\pi) \right\}$$

$\uparrow$ NOT 2π

admits a solution.

Indeed compactness follows from the estimate

$$\tfrac{1}{2}p^2 + \tfrac{1}{4}s^4 + \sin^6 x \cdot s \geq \tfrac{1}{2}p^2 + \tfrac{1}{8}s^4 - A$$

while LSC follows from the LSC of the norm for the term with $\dot{u}^2$, and from the uniform convergence for the terms with u.

Step 2 The min. point is unique because the Lagrangian is strictly convex in the pair (p,s).

Step 3 The min. point is of class C^∞ and satisfies the equation in classical sense in $[0,\pi]$ and the PBC $u(0) = u(\pi)$ and $\dot{u}(0) = \dot{u}(\pi)$. If we extend u to $\mathbb{R}$ by periodicity, we obtain a function of class C^2 (need to check that $\ddot{u}(0) = \ddot{u}(\pi)$) that solves the given equation (this needs to be checked) and therefore it is of class C^∞.

Step 4 Every periodic solution has a period T of the form $k\pi$ for some positive integer k.

Indeed, if u is T periodic, then also $\sin^6 x$ is T periodic.

Step 5 If $\bar{u}$ is a solution with period $k\pi$, then $\bar{u}$ is the same u that we found before. Indeed

- $\rightarrow$ $\bar{u}$ is a minimizer of the corresponding variational pbm in $[0,k\pi]$ (usual argument $F(\bar{u}+v) \geq F(\bar{u})$)
- $\rightarrow$ $\bar{u}$ is unique and solves (ELE) + PBC in $[0,k\pi]$
- $\rightarrow$ also u solves the same equation and is a minimizer to the same pbm.

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3. For every real number $r > 0$, let B_r denote the ball in the plane with center in the origin and radius r. For every positive integer m let us set

$$I(m,r) := \inf\left\{\int_{B_r}\left(|\nabla u|^{20} - |u_x \cdot u^m|\right)\,dx\,dy : u \in W_0^{1,20}(B_r)\right\}.$$

(a) Determine whether $I(1,r)$ is a real number.

(b) Determine all positive integers m for which $I(m,r)$ is a real number for every $r > 0$.

(c) Determine whether there exists admissible values of the parameters such that $I(m,r)$ is a negative real number.

(d) Determine whether $I(m,r) = 0$ for some admissible values of the parameters.

(a) $\boxed{\text{YES}}$

$$\int_{B_r}|\nabla u|^{20} - \int_{B_r}|u_x|\cdot|u| \underset{\text{Hölder}}{\geq} \int_{B_r}|\nabla u|^{20} - \left(\int_{B_r}u_x^2\right)^{1/2}\left(\int_{B_r}u^2\right)^{1/2}$$

$$\underset{\text{Poincaré}}{\geq} \int_{B_r}|\nabla u|^{20} - \left(\int_{B_r}|\nabla u|^2\right)^{1/2}\cdot \text{cost}\left(\int_{B_r}|\nabla u|^2\right)^{1/2}$$

$$= \int_{B_r}\left(|\nabla u|^{20} - \text{cost}\,|\nabla u|^2\right) \geq -\text{cost}\cdot\text{meas}(B_r)$$

$p^{20} - Ap^2 \geq -B \quad \forall p \in \mathbb{R}$

(b) $\boxed{m \leq 18}$ We distinguish 3 cases.

- If $m \leq 18$ we argue as in the case $m = 1$. The key point is that

$$\int_{B_r}|u_x|\cdot|u|^m \underset{\text{Hölder}}{\leq} \left(\int_{B_r}|u_x|^{20}\right)^{\frac{1}{20}}\left(\int_{B_r}|u|^{m\frac{20}{19}}\right)^{\frac{19}{20m}\cdot m}$$

$\frac{1}{20}$ $\frac{19}{20}$

$$\underset{\substack{\text{Sobolev–Poincaré} \\ \text{with } q=\frac{20}{19}m \text{ and } p=20}}{\leq} \left(\int_{B_r}|\nabla u|^{20}\right)^{\frac{1}{20}}\cdot\text{cost}\left(\int_{B_r}|\nabla u|^{20}\right)^{\frac{m}{20}}$$

- If $m \geq 20$ then $I(m,r) = -\infty$ for every $r > 0$. It is enough to consider $u(x,y) = \lambda\varphi(x,y)$ with $\varphi \in C_c^\infty(B_r)$ and let $\lambda \to +\infty$.
- If $m = 19$ then $I(m,r) = -\infty$ if r is large. If we consider $u(x,y) = \varphi\left(\frac{x}{r},\frac{y}{r}\right)$ with $\varphi \in C_c^\infty(B_1)$, then we discover that $F(u) < 0$ when r is large enough. Then consider λu with $\lambda \to +\infty$.

(c) $\boxed{\text{YES, when } m \leq 18 \text{ for any } r > 0}$ It is enough to consider any nontrivial $\varphi \in C_c^\infty(B_r)$ and observe that

$$F(\varepsilon\varphi) = \varepsilon^{20}\int|\nabla\varphi|^{20} - \varepsilon^{m+1}\int|\varphi_x|\cdot|\varphi| < 0 \quad \text{for } \varepsilon \text{ small.}$$

(d) $\boxed{\text{YES, when } m = 19 \text{ and } r \text{ is small}}$ The key idea is that, as in point (b) with $m \leq 18$, $\int_{B_r}|u_x|\cdot|u|^{19} \leq \text{const}(r)\int_{B_r}|\nabla u|^{20}$, and $\text{const}(r) \to 0$ as $r \to 0^+$. More precisely, actually $a = 19$

$\text{const}(r) = r^a\cdot\text{const}(1)$ for some $a > 0$ (usual homothety)

4. For every function $f : (0,2) \to \mathbb{R}$, let us set

$$[Tf](x) := \arctan(f(x)) \qquad \forall x \in (0,2).$$

Determine whether the restriction of T defines

(a) a strong-strong continuous operator $L^2((0,2)) \to L^2((0,2))$,

(b) a weak-weak continuous operator $L^2((0,2)) \to L^2((0,2))$,

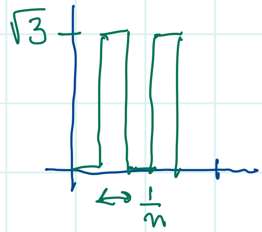
(c) a compact operator $H^1((0,2)) \to L^{20}((0,2))$,

(d) a compact operator $L^\infty((0,2)) \to L^2((0,2))$.

(a) $\boxed{\text{YES}}$ It is actually 1-Lip. continuous

$$\int_0^2 |\arctan(f) - \arctan(g)|^2 \underset{\substack{\uparrow \\ \arctan \text{ is } 1\text{-lip}}}{\leq} \int_0^2 |f-g|^2$$

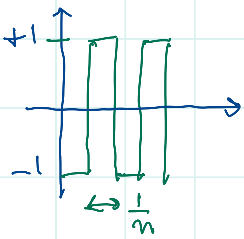
(b) $\boxed{\text{NO}}$ Consider f_n that alternates 0 and $\sqrt{3}$. Then $f_n \rightharpoonup \frac{\sqrt{3}}{2}$.



On the other hand, Tf_n alternates 0 and $\frac{\pi}{3}$, and therefore $Tf_n \rightharpoonup \underset{\substack{\shortparallel \\ \arctan\left(\frac{\sqrt{3}}{3}\right)}}{\frac{\pi}{6}} \neq \arctan\left(\frac{\sqrt{3}}{2}\right)$

(c) $\boxed{\text{YES}}$ If $\{f_n\}$ is bounded in H^1, then $\{f_n\}$ is equi $\frac{1}{2}$-Hölder, and therefore $\{\arctan(f_n)\}$ is equibounded in $L^\infty((0,2))$ and equi $\frac{1}{2}$-Hölder.

Ascoli-Arzelà thm $\Rightarrow$ there exists $f_{n_k} \to f_\infty$ unif. in $[0,2]$, and hence a fortiori $f_{n_k} \to f_\infty$ in $L^{20}((0,2))$



(d) $\boxed{\text{NO}}$ Consider f_n that alternates ± 1.

Then $\{f_n\}$ is clearly bounded in L^∞.

On the other hand, Tf_n alternates $\pm\frac{\pi}{4}$, and therefore $Tf_n \rightharpoonup 0$ weakly in L^2.

In particular, if $Tf_{n_k} \to g$ strongly in L^2, then necessarily $g \equiv 0$, and therefore $\underset{\substack{\shortparallel \\ \text{constant} \neq 0}}{\|Tf_{n_k}\|_{L^2}} \to 0$, which is clearly impossible.

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