Università di Pisa – Corso di Laurea in Matematica

Exam paper of “Istituzioni di Analisi Matematica”

Pisa, 01 July 2022

1. Let us consider the functional

$$F(u) = \int_0^1 \big(u'(x)^2 + xu'(x) + x^2u(x)\big)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 4$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 4 + u(1)$.

2. Discuss existence, uniqueness, and regularity of functions $u : \mathbb{R} \to \mathbb{R}$ that are periodic and satisfy

$$u'' = u^3 + \sin^6 x \qquad \forall x \in \mathbb{R}.$$

3. For every real number $r > 0$, let B_r denote the ball in the plane with center in the origin and radius r. For every positive integer m let us set

$$I(m, r) := \inf \left\{ \int_{B_r} \big(|\nabla u|^{20} - |u_x \cdot u^m|\big)\, dx\, dy : u \in W_0^{1,20}(B_r) \right\}.$$

   (a) Determine whether $I(1, r)$ is a real number.

   (b) Determine all positive integers m for which $I(m, r)$ is a real number for every $r > 0$.

   (c) Determine whether there exists admissible values of the parameters such that $I(m, r)$ is a negative real number.

   (d) Determine whether $I(m, r) = 0$ for some admissible values of the parameters.

4. For every function $f : (0, 2) \to \mathbb{R}$, let us set

$$[Tf](x) := \arctan(f(x)) \qquad \forall x \in (0, 2).$$

   Determine whether the restriction of T defines

   (a) a strong-strong continuous operator $L^2((0, 2)) \to L^2((0, 2))$,

   (b) a weak-weak continuous operator $L^2((0, 2)) \to L^2((0, 2))$,

   (c) a compact operator $H^1((0, 2)) \to L^{20}((0, 2))$,

   (d) a compact operator $L^\infty((0, 2)) \to L^2((0, 2))$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.