1. Let us consider the functional

$$F(u) = \int_0^\pi \left(u''(x)^2 + \sin x \cdot u(x)\right)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 5$.

(b) Discuss the minimum problem for $F(u)$ with conditions

$$u(0) = \int_0^\pi u(x)\, dx = 5.$$

(a) $\boxed{\text{Inf} = -\infty \text{ and min does not exist}}$

Just observe that $u_n(x) = 5 - nx$ satisfies the BC and $F(u_n) \to -\infty$

(b) $F(u+tv) = F(u) + t^2 \int_0^\pi \ddot{v}^2 + t\left\{ \int_0^\pi (u^{(4)} + \sin x)\, v(x) + [\ddot{u}\,\dot{v}]_0^\pi - [\dddot{u}\, v]_0^\pi \right\}$

if $v(0) = \int_0^\pi v(x)\,dx = 0.$

(ELE) $u^{(4)} = -\sin x + \lambda \quad \rightsquigarrow \quad u(x) = -\sin x + a + bx + cx^2 + dx^3 + ex^4$

$u(0) = 5, \quad \ddot{u}(0) = 0$

$\ddot{u}(\pi) = 0, \quad \dddot{u}(\pi) = 0$

$a = 5 \leftarrow u(0) = 5$, $\quad c = 0 \leftarrow \ddot{u}(0) = 0$, $\quad e = \frac{\lambda}{24}$

With the remaining three conditions we obtain a system of three equations in b, d, e:

$6\pi d + 12\pi^2 e = 0$

$6d + 24\pi e = 0$

$-\int_0^\pi u(x)\,dx = 0$

which has clearly a unique solution.

The corresponding solution u_0 is a min. point because

$$F(u_0+v) = F(u_0) + \underbrace{\delta F(u_0, v)}_{\substack{= \\ 0 \\ \text{because of ELE}}} + \underbrace{\int_0^\pi \ddot{v}(x)^2 dx}_{\substack{= \\ 0}} \iff \ddot{v} \equiv 0$$

$\iff v \equiv 0$ because $v(0) = 0$ and $\int v = 0$

Therefore the min. point is unique.

2. Let us consider, for every function $f \in L^\infty((0,1))$, the boundary value problem

$$u'' = u^5 + |f(x)| \cdot u, \qquad u(0) = 2022, \quad u'(1) = 0.$$

(a) Prove that the problem admits a unique solution.

(b) Discuss the regularity of this solution.

(c) Let $S : L^\infty((0,1)) \to L^\infty((0,1))$ be the operator that associates to each function f the corresponding solution u. Determine whether S is a compact operator.

(a) Standard direct method with variational formulation

$$\min \left\{ \int_0^1 \underbrace{\left(\frac{1}{2} \dot{u}^2 + \frac{1}{6} u^6 + \frac{1}{2} |f(x)| u^2 \right)}_{L(x,s,p)} : u \in H^1((0,1)),\ u(0) = 2022 \right\}$$

Compactness follows from $L(x,s,p) \geq \frac{1}{2} p^2 + \frac{1}{6} s^6$

(here the absolute value is essential)

SCI is standard

The NBC in $x = 1$ appears in the min. process.

(b) The minimum is of class $C^{1,1}$ (namely $\dot{u}$ is Lipschitz).

In the usual way we discover that

$$(\dot{u})' = u^5 + |f(x)| \cdot u$$

weak deriv. of $\dot{u}$

$$u \in H^1 \Rightarrow \text{RHS} \in L^\infty \Rightarrow \dot{u} \in W^{1,\infty} \Rightarrow u \in W^{2,\infty} = C^{1,1}$$

(c) $\boxed{\text{YES}}$ If $\{f_n\} \subseteq L^\infty((0,1))$ is bounded, then min. values are bounded ($u \equiv 5$ is always a competitor)

It follows that $\{\dot{u}_n\} \subseteq L^2((0,1))$ is bounded (usual estimate $L(x,s,p) \geq \frac{1}{2} p^2$) and hence also $\{u_n\} \subseteq L^\infty((0,1))$ is bounded (BC). It follows that

$$u_{n_k} \longrightarrow u_\infty \quad \text{uniformly (Ascoli – Arzelà)}$$

and in particular in $L^\infty((0,1))$

Rmk It can be checked that u_∞ is the solution corresponding to f_∞, where f_∞ is the weak limit of f_n (which exists along the same subsequence).

3. For every positive integer d, let B_d denote the unit ball in $\mathbb{R}^d$. For every real number $M > 0$, let us set

$$\mathcal{S}(d, M) := \sup \left\{ \int_{B_d} \arctan(u^{20}(x))\, dx : u \in H_0^1(B_d),\ \int_{B_d} |\nabla u(x)|^2\, dx \le M \right\}.$$

(a) Determine whether the supremum is actually a maximum.

(b) Determine the following limits

$$\lim_{M \to +\infty} \mathcal{S}(d, M), \qquad \lim_{M \to 0^+} \mathcal{S}(d, M).$$

(a) $\boxed{\text{The supremum is ALWAYS a maximum}}$

Standard direct method:

$|\nabla u_n|$ is bounded in L^2 by assumption
u_n is bounded in L^2 due to BC (Poincaré) $\Big\} \Rightarrow$ compactness

$\int \arctan(u^{20})$ is continuous wrt L^2 convergence.

(b) $\boxed{\lim_{M \to +\infty} S(d, M) = \frac{\pi}{2} \cdot \text{meas}(B_d)}$ Indeed, for every $R \in (0,1)$ and every $K > 0$ there exists $u \in C_c^\infty(B_d)$ s.t. $u(x) \ge K$ if $|x| \le R$.

This function is a competitor in the definition of $S(d, M)$ when M is large enough.

(c) $\boxed{\lim_{M \to 0^+} S(d, M) = 0}$ Indeed, let us observe that there exists a constant c s.t. $\arctan(x^{20}) \le c x^2$ for every $x \in \mathbb{R}$. is it enough?

Let u_n be a minimizer for $S(d, \frac{1}{n})$. Then

$$u_n \to 0 \quad \text{in } L^2(B_d) \qquad \text{(Poincaré)}$$

and therefore

$$\int \arctan(u_n^{20}) \le c \int u_n^2 \to 0$$

Alternative: $u_n \to 0$ in $L^2(B_d) \Rightarrow u_{n_k} \to 0$ for a.e. $x \in B_d$
↑ is it enough?

$$\Rightarrow \int_{B_d} \arctan(u_{n_k}^{20})\, dx \to 0 \quad \text{(dominated convergence)}$$

4. For every function $f \in L^1_{loc}(\mathbb{R})$, let us set

$$[Tf](x) := \sin\left(\int_0^x f(t)\,dt\right) \qquad \forall x \in \mathbb{R}.$$

Determine whether the restriction of T defines

(a) a Lipschitz continuous operator $L^4((0,1)) \to L^7((0,1))$,

(b) a compact operator $L^2((0,1)) \to L^\infty((0,1))$,

(c) a continuous operator $L^\infty((0,+\infty)) \to L^\infty((0,+\infty))$.

(a) Set $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_0^x g(t)\,dt$.

$$|\sin(F(x)) - \sin(G(x))| \le |F(x) - G(x)| \le \int_0^x |f(t)-g(t)|_{\frac{1}{4}} \cdot 1_{\frac{3}{4}}\,dt$$

$$\underset{\text{Hölder}}{\le} \|f-g\|_{L^4} \cdot x^{3/4} \le \|f-g\|_{L^4} \qquad \forall x \in (0,1).$$

This proves that $T : L^4 \to L^\infty$ is Lipschitz.
A fortiori $T : L^4 \to L^7$ is Lipschitz.

(b) $\boxed{\text{YES}}$ $\{f_n\} \subseteq L^2$ bounded $\Rightarrow$ F_n are equi $\frac{1}{2}$-Hölder cont.
and $F_n(0) = 0$

$\Rightarrow F_{n_k} \to F_\infty$ unif. (Ascoli - Arzelà)

$\Rightarrow \sin(F_{n_k}) \to \sin(F_\infty)$ unif., and hence in $L^\infty((0,1))$.

(c) $\boxed{\text{NO}}$ Consider $f_n(x) \equiv \frac{1}{n}$, so that $f_n \to 0$ in $L^\infty((0,+\infty))$
but

$$[Tf_n](x) = \sin\left(\frac{x}{n}\right) \not\to 0 \quad \text{in } L^\infty((0,+\infty)).$$

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