Università di Pisa – Corso di Laurea in Matematica

# Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 10 June 2022

1. Let us consider the functional
$$F(u) = \int_0^\pi \left(u''(x)^2 + \sin x \cdot u(x)\right)\,dx.$$
   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 5$.
   (b) Discuss the minimum problem for $F(u)$ with conditions
$$u(0) = \int_0^\pi u(x)\,dx = 5.$$

2. Let us consider, for every function $f \in L^\infty((0,1))$, the boundary value problem
$$u'' = u^5 + |f(x)| \cdot u, \qquad u(0) = 2022, \quad u'(1) = 0.$$
   (a) Prove that the problem admits a unique solution.
   (b) Discuss the regularity of this solution.
   (c) Let $S : L^\infty((0,1)) \to L^\infty((0,1))$ be the operator that associates to each function $f$ the corresponding solution $u$. Determine whether $S$ is a compact operator.

3. For every positive integer $d$, let $B_d$ denote the unit ball in $\mathbb{R}^d$. For every real number $M > 0$, let us set
$$\mathcal{S}(d,M) := \sup\left\{\int_{B_d} \arctan(u^{20}(x))\,dx : u \in H_0^1(B_d),\ \int_{B_d} |\nabla u(x)|^2\,dx \leq M\right\}.$$
   (a) Determine whether the supremum is actually a maximum.
   (b) Determine the following limits
$$\lim_{M\to+\infty} \mathcal{S}(d,M), \qquad \lim_{M\to 0^+} \mathcal{S}(d,M).$$

4. For every function $f \in L^1_{loc}(\mathbb{R})$, let us set
$$[Tf](x) := \sin\left(\int_0^x f(t)\,dt\right) \qquad \forall x \in \mathbb{R}.$$
   Determine whether the restriction of $T$ defines
   (a) a Lipschitz continuous operator $L^4((0,1)) \to L^7((0,1))$,
   (b) a compact operator $L^2((0,1)) \to L^\infty((0,1))$,
   (c) a continuous operator $L^\infty((0,+\infty)) \to L^\infty((0,+\infty))$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.