1. Let us consider the minimum problem

$$\min\left\{\int_0^1 \left[(u'-x)^2+(u-x^2)^2\right]\,dx : u\in C^1([0,1]),\ u(0)=a\right\}.$$

(a) Determine for which values of the real parameter a the problem admits a unique solution.

(b) Determine for which values of the real parameter a the solution is a polynomial.

ELE $\quad (L_p)' = L_s \qquad (\dot{u}-x)' = u - x^2 \rightsquigarrow \ddot{u} = u+1-x^2$

NBC $\quad L_p = 0 \rightsquigarrow \dot{u}(1) = 1$

$$\begin{cases} \ddot{u} = u+1-x^2 \\ u(0)=a \\ \dot{u}(1)=1 \end{cases} \quad \rightsquigarrow \quad \begin{aligned} & u(x) = x^2+1+\alpha\sinh x+\beta\cosh x \\ & \beta+1 = a \\ & 2+\alpha\cosh 1+\beta\sinh 1 = 1 \end{aligned} \quad \rightsquigarrow \text{unique solution } (\alpha_0,\beta_0)$$

CLAIM: $u_0(x) = x^2+1+\alpha_0\sinh x+\beta_0\cosh x$ is the unique minimizer.

Indeed any other competitor $w(x)$ can be written as $w(x) = u_0(x)+v(x)$ with $v(0)=0$ and

$$F(w) = F(u_0) + \underbrace{\delta F(u_0,v)}_{=0} + \int_0^1 (\dot{v}^2+v^2)\,dx$$

$0 \rightarrow$ this needs to be checked

$\geq F(u_0)$ with equality if and only if $v(x)\equiv 0$.

The solution is a polynomial if and only if $\alpha_0=\beta_0=0$, but this is impossible because of the NBC.

Conclusion: $\boxed{\begin{array}{l}\rightarrow \text{the minimizer exists and is unique for every } a\in\mathbb{R} \\ \rightarrow \text{the minimizer is NEVER a polynomial.}\end{array}}$

2. Discuss existence, uniqueness, and regularity of solutions to the boundary value problem

$$u'' = \frac{u+|x|}{1+|u'|}, \qquad u(-1)=3, \quad u(1)=4.$$

$\ddot{u} + |\dot{u}|\ddot{u} = u + |x| \rightsquigarrow$ ELE of

$$\min\left\{ \int_{-1}^{1} \left(\frac{1}{2}\dot{u}^2 + \frac{1}{6}|\dot{u}|^3 + \frac{1}{2}u^2 + |x|u\right)dx : u \in H^1((-1,1)) + \text{DBC} \right\}$$

$L(x,s,p) = \frac{1}{2}p^2 + \frac{1}{6}|p|^3 + \frac{1}{2}s^2 + |x|s \quad \rightsquigarrow \quad L_p = p + \frac{1}{2}|p|p =: \varphi(p)$

Standard direct method

$\to$ weak formulation with $F: H^1((-1,1)) \to \mathbb{R} \cup \{+\infty\}$ NEEDED

$\to$ compactness because $L(x,s,p) \geq \frac{1}{2}p^2 + \frac{1}{2}s^2 - |s| \geq \frac{1}{2}p^2 - A$ + DBC

$\to$ LSC because $p \to \frac{1}{2}p^2 + \frac{1}{6}|p|^3$ is convex and bounded from below

with respect to $u_n \to u_\infty$ unif. and $\dot{u}_n \to \dot{u}_\infty$ L^2-weak.

$\to$ Regularity: in the usual way we find that

[Details needed]

$$\Big(\underbrace{\dot{u} + \tfrac{1}{2}|\dot{u}|\dot{u}}_{}\Big)' = u + |x|$$

$\varphi(\dot{u})$ with $\varphi(p)$ function of class C^1 with inverse of class C^1

$u \in H^1 \Rightarrow u \in C^0 \Rightarrow \text{RHS} \in C^0$

$\Rightarrow \varphi(\dot{u}) \in C^1 \Rightarrow \dot{u} \in C^1$

$\varphi^{-1} \in C^1$

$\Rightarrow u \in C^2 \;\; \not\Rightarrow \;\; \boxed{u \in C^{2,1}}$ at least, but actually optimal

$\to$ Uniqueness: thanks to the convexity of $p \to |p|^3$ we find that, for every $v \in H^1_0((-1,1))$:

$$F(u+v) = F(u) + \underset{0}{\delta F(u,v)} + \frac{1}{2}\int_{-1}^{1}(\dot{v}^2 + v^2)\,dx$$

if u solves the given equation

$\geq F(u)$ with equality if and only if $v \equiv 0$

As a consequence

$\to$ every solution to the equation is a minimizer

$\to$ the minimizer is unique.

3. Let $\Omega := (-1,1)^2$ denote a square in the plane. Let us consider the set

$$\mathcal{S}(p) := \left\{ u \in C^1(\Omega) : \int_\Omega u(x,y)\,dx\,dy = 20, \quad \int_\Omega (|u_x(x,y)|^p + |u_y(x,y)|^p)\,dx\,dy \le 22 \right\},$$

Determine for which values of the real exponent $p \ge 1$ the following three quantities are finite:

$$C_1(p) := \sup\{u(0,0) : u \in \mathcal{S}(p)\}, \qquad C_2(p) := \inf\left\{ \int_\Omega u(x,y)^3\,dx\,dy : u \in \mathcal{S}(p) \right\},$$

$$C_3(p) := \sup\left\{ \int_{-1}^1 u(t,t)^8\,dt : u \in \mathcal{S}(p) \right\}.$$

General idea:

$\to C_1(p)$ is finite $\iff W^{1,p}(\Omega) \to L^\infty(\Omega) \iff \boxed{p>2}$

$\to C_2(p)$ is finite $\iff W^{1,p}(\Omega) \to L^3(\Omega) \iff 3 \le p_* \iff \boxed{p \ge \frac{6}{5}}$

$\to C_3(p)$ is finite $\iff \underset{\uparrow \text{trace}}{Tr}: W^{1,p}(\Omega) \to L^8(\text{diagonal}) \iff 8 \le \tilde{p}_* \iff \boxed{p \ge \frac{16}{9}}$

[To be precise, this is true in every smaller cube]

Positive part: $\|u - u_\Omega\|_p \le C \|\nabla u\|_p$

u_Ω: FIXED; $\le$: PSW; $\|\nabla u\|_p$: BOUNDED in $S(p)$

Therefore there exists $M \in \mathbb{R}$ s.t. $\|u\|_{1,p,\Omega} \le M \quad \forall u \in S(p)$

Counterexamples

$\boxed{C_1(p)}$ Take any $u \in H^1(\Omega)$ but unbounded in $(0,0)$. We can assume that $\|\nabla u\|_{L^2} \le 1$ and $\int u = 20$

if not, consider λu if not, consider $u + c$

Now approximate in H^1 with $u_n \in C^1$ and adjust to preserve $\int u_n = 20$

$\boxed{C_2(p)}$ Take $\varphi \in C_c^\infty(\Omega)$ with $\varphi \ge 0$ and $\varphi \not\equiv 0$. Define $u_n(x) := -a_n \varphi(nx)$

If $3 > p_*$ we can choose a_n such that

$\|\nabla u_n\|_{L^p} = \text{constant}, \quad \int_\Omega u_n^3 \to -\infty, \quad \int_\Omega u_n \to 0$

Now add a constant in order to adjust $(u_n)_\Omega$

$\boxed{C_3(p)}$ Same as before, of course with $\varphi \not\equiv 0$ on the diagonal of the square.

4. Let us consider the operator $T : L^2((0,1)) \to L^2((0,\pi))$ defined by

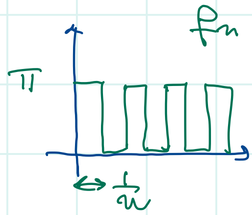
$$[Tf](x) := \int_0^{\sin x} \sin(f(t))\,dt \qquad \forall x \in (0,\pi).$$

Determine whether the operator T is

(a) linear,
(b) strong-strong continuous,
(c) weak-strong continuous,
(d) compact,
(e) Lipschitz continuous (if this is the case, then find suitable bounds on the Lipschitz constant).

(a) T is $\boxed{\text{NOT}}$ linear. Take $f(x) \equiv \pi$ and observe that $Tf \equiv 0$ but $T(\frac{1}{2}f) \not\equiv 0$.

(b) $\boxed{\text{YES}}$. If $f_n \xrightarrow{L^2} f_\infty$, then up to subsequences $f_n(x) \to f_\infty(x)$ almost everywhere, then $\sin(f_n(t)) \to \sin(f_\infty(t))$ a.e. with equibound $\Rightarrow [Tf_n](x) \to [Tf_\infty](x)$ for every x with equi-bound $\Rightarrow Tf_n \to Tf_\infty$ in L^2.



(c) $\boxed{\text{NO}}$ Consider $f_n(x)$ that alternates 0 and π. Then $f_n \rightharpoonup f_\infty \equiv \frac{\pi}{2}$. On the other hand $Tf_n \equiv 0$ but $Tf_\infty \neq 0$.

(d) $\boxed{\text{YES}}$ $\{f_n\}$ bounded in $L^2((0,1)) \Rightarrow \{Tf_n\}$ bounded in $L^\infty((0,\pi))$ and 1-lip. continuous $\underset{\text{A.A.}}{\Rightarrow}$ $\{f_n\}$ rel. cpt. in $C^0([0,\pi])$ $\Rightarrow \{f_n\}$ rel. cpt. in $L^2((0,\pi))$.

(e) $$|[Tf](x) - [Tg](x)| \le \int_0^{\sin x} |\sin(f(t)) - \sin(g(t))|\,dt$$
$$\le \int_0^{\sin x} |f(t)-g(t)|\,dt \le |\sin x|^{1/2} \|f-g\|_{L^2((0,1))}$$

$$\Rightarrow \|Tf - Tg\|_{L^2((0,\pi))} \le \|f-g\|_{L^2} \left\{\int_0^\pi \sin x\,dx\right\}^{1/2} \le \sqrt{2}\,\|f-g\|_{L^2} \Rightarrow \boxed{L \le \sqrt{2}}$$

Now take $g(x) \equiv 0$ and $f(x) \equiv \frac{1}{n}$. Then

$$L \cdot \frac{1}{n} = L\,\|f-g\|_{L^2((0,1))} \ge \|Tf - Tg\|_{L^2((0,\pi))} = \|\sin\tfrac{1}{n} \cdot \sin x\|_{L^2((0,\pi))}$$
$$= \sin\tfrac{1}{n} \cdot \sqrt{\tfrac{\pi}{2}} \quad \Rightarrow \quad \boxed{L \ge \sqrt{\tfrac{\pi}{2}}}$$

Appendix

$\boxed{1}$ In pbm 2 the solution is of class $C^{2,1}$ but not of class C^3. Indeed, from

$$(|\dot{u}|+1)\,\ddot{u} - u = |x|$$

we see that the LHS is NOT differentiable in $x=0$. If $u \in C^3$, this is possible only if $\dot{u}(0)=0$. If $u(0)=c$, then by uniqueness we deduce that the solution to the <u>CAUCHY PBM</u>

$$\ddot{u} = \frac{u+|x|}{1+|\dot{u}|}$$

is an <u>even</u> function, which is not compatible with the DBCs.

$\boxed{2}$ It should be possible to prove that the Lipschitz constant in pbm (4e) is the solution to the following variational problem (which reminds us of the Poincaré constant with 1 DBC)

$$\max\left\{ \int_0^\pi u(\sin x)^2\,dx \ : \ \int_0^1 \dot{u}(x)^2\,dx = 1,\ u(0)=0 \right\}$$

(actually the constant is the square root of the max, whose existence should be an application of the direct method)

$\boxed{\le}$ Given f and g, set $\dot{u}(t) = |f(t)-g(t)|$. Then

$$|[Tf](x) - [Tg](x)| \le \int_0^{\sin x} \underbrace{|f(t)-g(t)|}_{\dot{u}(t)}\,dt = u(\sin x)$$

and therefore

$$\|Tf-Tg\|_{L^2((0,\pi))} = \|u(\sin x)\|_{L^2((0,\pi))} \le L\,\|\dot{u}(x)\|_{L^2((0,1))} = L\,\|f-g\|_{L^2((0,1))}$$

$\boxed{\ge}$ Consider $g(x)\equiv 0$ and $f(x) = \frac{1}{m}\dot{u}_0(x)$, where $u_0(x)$ is a maximizer for the variational problem.