Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 11 January 2022

1. Let us consider the functional

$$F(u) = \int_0^1 \left(u'(x)^2 + x^2 u'(x) + u(x)^2 \right) dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = 0$.

   (b) Determine whether there exists a real number c such that the minimizer of $F(u)$ subject to the integral constraint

$$\int_0^1 u(x)\, dx = c$$

   is a constant function.

2. Discuss existence, uniqueness, and regularity of solutions to the boundary value problem

$$(2 + e^{u'}) \cdot u'' = u^3 - e^{3x}, \qquad u(0) = u(3) = 1.$$

3. Let B denote a ball in the space $\mathbb{R}^3$. For every $p \geq 1$ we consider the set

$$\mathcal{S}(B,p) := \left\{ u \in C_c^\infty(B) : \int_B \left(|\nabla u(x)|^p - u(x)^2 \right) dx \leq 2 \right\}.$$

   (a) Determine whether the set $\mathcal{S}(B,1)$ is bounded in $L^1(B)$.

   (b) Determine for which values of p the set $\mathcal{S}(B,p)$ is relatively compact in $L^7(B)$.

   (c) Determine for which values of p and q the set $\mathcal{S}(B,p)$ is relatively compact in $L^q(B)$.

4. Determine whether there exists a function $f : (-8, 8) \to \mathbb{R}$ of class C^8 such that

$$f(x) = \cos x + \int_0^{\cos x} \cos(f(t))\, dt \qquad \forall x \in (-8, 8).$$

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.