Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 01 February 2022

1. Let us consider the minimum problem

$$\min\left\{\int_0^1 \left[(u'-x)^2+(u-x^2)^2\right]\,dx : u\in C^1([0,1]),\ u(0)=a\right\}.$$

   (a) Determine for which values of the real parameter a the problem admits a unique solution.

   (b) Determine for which values of the real parameter a the solution is a polynomial.

2. Discuss existence, uniqueness, and regularity of solutions to the boundary value problem

$$u''=\frac{u+|x|}{1+|u'|},\qquad u(-1)=3,\quad u(1)=4.$$

3. Let $\Omega:=(-1,1)^2$ denote a square in the plane. Let us consider the set

$$\mathcal{S}(p):=\left\{u\in C^1(\Omega):\int_\Omega u(x,y)\,dx\,dy=20,\quad \int_\Omega \left(|u_x(x,y)|^p+|u_y(x,y)|^p\right)\,dx\,dy\le 22\right\},$$

   Determine for which values of the real exponent $p\ge 1$ the following three quantities are finite:

$$C_1(p):=\sup\{u(0,0):u\in\mathcal{S}(p)\},\qquad C_2(p):=\inf\left\{\int_\Omega u(x,y)^3\,dx\,dy:u\in\mathcal{S}(p)\right\},$$

$$C_3(p):=\sup\left\{\int_{-1}^1 u(t,t)^8\,dt:u\in\mathcal{S}(p)\right\}.$$

4. Let us consider the operator $T:L^2((0,1))\to L^2((0,\pi))$ defined by

$$[Tf](x):=\int_0^{\sin x}\sin(f(t))\,dt\qquad \forall x\in(0,\pi).$$

   Determine whether the operator T is

   (a) linear,

   (b) strong-strong continuous,

   (c) weak-strong continuous,

   (d) compact,

   (e) Lipschitz continuous (if this is the case, then find suitable bounds, from above and from below, for the Lipschitz constant).

> Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.

Course IstAM_22 by Massimo Gobbino – **Exam paper 2022_2**