Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 11 January 2021

1. Let us consider the functional

$$F(u) = \int_0^1 \left(u'(x)^2 + x^2 u'(x) + u(x)^2 \right) \, dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = 0$.

   (b) Determine whether there exists a real number c such that the minimizer of $F(u)$ subject to the integral constraint

$$\int_0^1 u(x) \, dx = c$$

   is a constant function.

2. Discuss existence, uniqueness, and regularity of solutions to the boundary value problem

$$(2 + e^{u'}) \cdot u'' = u^3 - e^{3x}, \qquad u(0) = u(3) = 1.$$

3. Let B denote a ball in the space $\mathbb{R}^3$. For every $p \geq 1$ we consider the set

$$\mathcal{S}(B, p) := \left\{ u \in C_c^\infty(B) : \int_B \left(|\nabla u(x)|^p - u(x)^2 \right) \, dx \leq 2 \right\}.$$

   (a) Determine whether the set $\mathcal{S}(B, 1)$ is bounded in $L^1(B)$.

   (b) Determine for which values of p the set $\mathcal{S}(B, p)$ is relatively compact in $L^7(B)$.

   (c) Determine for which values of p and q the set $\mathcal{S}(B, p)$ is relatively compact in $L^q(B)$.

4. Determine whether there exists a function $f : (-8, 8) \to \mathbb{R}$ of class C^8 such that

$$f(x) = \cos x + \int_0^{\cos x} \cos(f(t)) \, dt \qquad \forall x \in (-8, 8).$$

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.

1. Let us consider the functional

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(a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = 0$.

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is a constant function.

(a) $L(x,s,p) = p^2 + x^2 p + s^2$ $L_p = (2p + x^2)$

ELE: $(L_p)' = L_s \rightsquigarrow (2\dot{u} + x^2)' = 2u \rightsquigarrow \ddot{u} = u - x$

NBC: $2\dot{u}(1) + 1 = 0 \rightsquigarrow \dot{u}(1) = -\frac{1}{2}$

$$\begin{cases} \ddot{u} = u - x & \rightsquigarrow u(x) = x + a\sinh x + b\cosh x \\ u(0) = 0 & \rightsquigarrow b = 0 \\ \dot{u}(1) = -\frac{1}{2} & \rightsquigarrow 1 + a\cosh 1 = -\frac{1}{2} \rightsquigarrow \text{find a unique } a \end{cases}$$

The unique solution $u_0(x)$ is the unique min. point. Any other competitor $w(x)$ can be written as $w(x) = u_0(x) + v(x)$, with $v(0) = 0$.

Usual computation ...

$$F(w) = F(u_0) + \underset{\overset{\shortparallel}{0}}{\delta F(u_0, v)} + \int_0^1 \dot{v}(x)^2 dx + \int_0^1 v(x)^2 dx$$

$\geq F(u_0)$ with $=$ if and only if $v(x) \equiv 0$.

(b) Assume that $u_0(x)$ is a minimizer. Then with FLCV with 0 average (or Lagrange multipliers) we find that

$$\ddot{u}_0 = u_0 - x + \lambda$$

It is trivial that no constant function satisfies this equation.

NOTE ① One should be more precise about the regularity of u_0 that is required for the previous equ.

② Existence of minimizers (not required in point (b)) can be proved by either indirect or direct method.

2. Discuss existence, uniqueness, and regularity of solutions to the boundary value problem

$$(2+e^{u'})\cdot u'' = u^3 - e^{3x}, \qquad u(0)=u(3)=1.$$

Variational formulation $\quad F(u) := \int_0^3 \left(\dot{u}^2 + e^{\dot{u}} + \frac{1}{4}u^4 - e^{3x}u\right)dx$

$$\min\{F(u) : u \in H^1((0,3)),\ u(0)=u(3)=1\}$$

↑ $F : H^1((0,3)) \to \mathbb{R}\cup\{+\infty\}$ ↑ well-defined in H^1

Standard DIRECT METHOD.

→ Compactness follows from the estimate

$$L(x,s,p) = p^2 + e^p + \frac{1}{4}s^4 - e^{3x}s \geq p^2 + \underbrace{\frac{1}{4}s^4 - e^9|s|}_{\text{bounded from below}} \geq p^2 - A$$

and from the DBC

→ LSC: if $u_n \to u_\infty$ uniformly and $\dot{u}_n \to \dot{u}_\infty$ weakly in L^2, then

$$\liminf \int_0^3 (\dot{u}_n^2 + e^{\dot{u}_n}) \geq \int_0^3 (\dot{u}_\infty^2 + e^{\dot{u}_\infty}) \quad \left(\begin{array}{l} p \to p^2+e^p \text{ is convex and} \\ \text{bounded from below}\end{array}\right)$$

$$\lim \int_0^3 \left(\frac{1}{4}u_n^4 - e^{3x}u_n\right) = \int_0^3 \left(\frac{1}{4}u_\infty^4 - e^{3x}u_\infty\right) \quad \text{(unif. convergence)}$$

→ Regularity: Let $u_0 \in H^1((0,3))$ be any minimizer. Define $\varphi(t) := F(u_0 + tv)$ with $v \in C_c^\infty((0,3))$ and find that

$$0 = \varphi'(0) = \int_0^3 (2\dot{u}_0 + e^{\dot{u}_0})\dot{v} + (u_0^3 - e^{3x})v$$

↑ One NEEDS to verify that one can differentiate the integral !!!
This requires that $e^{\dot{u}_0} \in L^1$, which follows from $F(u_0) < +\infty$

This proves that $(2\dot{u}_0 + e^{\dot{u}_0})' = u_0^3 - e^{3x}$ from which we proceed in the usual way (we used that $p \to 2p + e^p$ has an inverse of class C^∞)

→ Uniqueness. Exploiting convexity we obtain that

$$F(u_0+v) \geq F(u_0) + \underset{\substack{\| \\ 0}}{\delta F(u_0, v)} + \int_0^3 \dot{v}_0(x)^2 \geq F(u_0) \text{ with equality}$$

if and only if ...

3. Let B denote a ball in the space $\mathbb{R}^3$. For every $p \geq 1$ we consider the set

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(a) Determine whether the set $\mathcal{S}(B,1)$ is bounded in $L^1(B)$.

(b) Determine for which values of p the set $\mathcal{S}(B,p)$ is relatively compact in $L^7(B)$.

(c) Determine for which values of p and q the set $\mathcal{S}(B,p)$ is relatively compact in $L^q(B)$.

Step 1 If $p<2$, then $S(B,p)$ is NOT bounded in any $L^q(B)$.
Indeed, consider $\varphi \in C_c^\infty(B)$ with $\varphi(x) \not\equiv 0$, and define $u_n(x) = n\varphi(x)$.
Then $u_n \in S(B,p)$ when n is large and $\|u_n\|_{L^q(B)} \to +\infty$.

Step 2 If $p>2$, then $S(B,p)$ is bounded in $W^{1,p}(B)$, and hence relatively compact in $L^q(B)$ for every $q < p_*$ (with the agreement that $p_* = +\infty$ when $p \geq 3$). If $q=7$, this is true if $p > \frac{21}{10}$.
Indeed, from Poincaré-Sobolev inequality $\|u\|_{L^2} \leq C(p)\|\nabla u\|_{L^p}$, so that

$$2 \geq \int_B (|\nabla u|^p - u^2) \geq \|\nabla u\|_{L^p}^p - C(p)^2 \|\nabla u\|_{L^p}^2 \rightsquigarrow \|\nabla u\|_{L^p} \text{ bounded}$$

$\rightsquigarrow \|u\|_{1,p,B}$ bounded
↑ Poincaré again

Step 3 If $q \geq p_*$, then $S(B,p)$ is NOT relatively compact in $L^q(B)$.
Indeed, the counterexample to the compact embedding $W^{1,p} \to L^{p_*}$ delivers us a sequence $\{u_n\} \subseteq C_c^\infty(B)$ such that

$\|u_n\|_{1,p,B} = 1$ — this implies $\{u_n\} \subseteq S(B,p)$

$\|u_n - u_m\|_{L^{p_*}} \geq c_0 > 0$ if $n \neq m$
↑ a fortiori this is true in L^q if $q \geq p_*$

Step 4 If $p=2$, the answer depends on the Poincaré constant P_B of B, which is proportional to R^2 (R = radius of B).
→ If $P_B < 1$ (small radius), the same argument as before shows that $S(B,2)$ is bounded in $W^{1,2}(B)$, and hence rel. cpt. in $L^p(B)$ iff $q<6$.
→ If $P_B \geq 1$ (large radius), then there exists $\varphi \in H_0^1(B)$ with $\varphi \not\equiv 0$ s.t. $\int(|\nabla\varphi|^2 - \varphi^2) = 0$. Now consider $u_n = n\varphi$. Then again $\int(|\nabla u_n|^2 - u_n^2) = 0$ and $\|u_n\|_{L^1}$ very large. By approx we find $\tilde{u}_n$ s.t. $\int(|\nabla \tilde{u}_n|^2 - \tilde{u}_n^2) \leq 2$ and $\|\tilde{u}_n\|_{L^1}$ very large
↑ a fortiori in L^q

4. Determine whether there exists a function $f : (-8, 8) \to \mathbb{R}$ of class C^8 such that

$$f(x) = \cos x + \int_0^{\cos x} \cos(f(t))\,dt \qquad \forall x \in (-8, 8).$$

Consider the triple $V = C \supseteq K$ with
(V normed, C convex, K compact — cpt. thanks to Ascoli–Arzelà)

$V := C^0([-8,8])$ wrt the sup norm

$K := \{u \in V : \|u\|_{L^\infty} \leq 2 \text{ and } u \text{ is 2-lip. continuous}\}$

Consider the operator $(Tf)(x) := \cos x + \int_0^{\cos x} \cos(f(t))\,dt$

We claim that $T : C \to K$ and is continuous

→ Continuity: observe that

$$|[Tf](x) - [Tg](x)| \leq \left| \int_0^{\cos x} (\cos(f(t)) - \cos(g(t)))\,dt \right|$$
$$\leq \int_{-1}^{1} |f(t) - g(t)|\,dt \leq 2\|f-g\|_{L^\infty}$$

so that T is actually Lipschitz continuous

→ Image$(T) \subseteq K$. The bound $|[Tf](x)| \leq 2$ is almost trivial, while the 2-lip. continuity follows from

$$|[Tf](x_2) - [Tf](x_1)| \leq |\cos x_2 - \cos x_1| + \Big| \int_{\cos x_1}^{\cos x_2} \underbrace{\cos(f(t))}_{\leq 1}\,dt \Big| \leq 2|\cos x_2 - \cos x_1|.$$

SCHAUDER FIXED POINT THM $\Rightarrow$ there exists $f \in C^0([-8,8])$ s.t. $Tf = f$, namely s.t. f solves the given equation.

BOOTSTRAP ARGUMENT $\Rightarrow$ any fixed point in C^0 is actually in C^∞.

Remark We can set the fixed point argument in any L^p space instead of C^0. Setting directly in C^8 is quite annoying !!!