Forme canoniche 3

Argomenti: forme canoniche di applicazioni lineari **Difficoltà:** $\star\star\star$

Prerequisiti: autovalori, autovettori, forme canoniche, matrici di cambio di base

1. Determinare, per ciascuna delle seguenti matrici, la forma canonica reale e quella complessa (se diversa da quella reale). Determinare anche, nel caso reale ed eventualmente nel caso complesso, una possibile matrice di cambio di base che realizza il passaggio alla forma canonica.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -5 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 6 & 0 & 1 \\ 2 & 5 & 2 \\ -1 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 \\ -6 & 5 & 6 \\ 6 & -3 & -4 \end{pmatrix} \qquad \begin{pmatrix} -1 & -2 & -1 \\ 2 & 3 & 1 \\ 4 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ 3 & 3 & 3 \end{pmatrix} \qquad \begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ -3 & -3 & -3 \end{pmatrix}$$

2. Determinare per quali valori del parametro a la matrice $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & a \\ 0 & 0 & 1 \end{pmatrix}$ è diagonalizzabile.

3. Determinare per quale valore del parametro reale a le due matrici

$$\begin{pmatrix} 1 & 7 & 5 \\ 0 & 2 & -2 \\ 0 & 0 & a \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & a \end{pmatrix}$$

sono simili.

4. Consideriamo le seguenti applicazioni lineari $f : \mathbb{R}^3 \to \mathbb{R}^3$:

$$(x, y, z) \to (y, z, z), \qquad (x, y, z) \to (x, x + y, x + y + z).$$

Per ciascuna di essere determinare la forma canonica, ed una base in cui la matrice assume la forma canonica.

5. Consideriamo le seguenti applicazioni lineari $f : \mathbb{R}_{\leq 2}[x] \to \mathbb{R}_{\leq 2}[x]$:

$$p(x) \to p(-x), \qquad p(x) \to p(x) + p(-x), \qquad p(x) \to p(x) + p(2x),$$

$$p(x) \to p'(x), \qquad p(x) \to p(x) + p(x + 1), \qquad p(x) \to p(7) \cdot x^2,$$

$$p(x) \to xp'(x) + 2p(x), \qquad p(x) \to (x + 2)p'(x), \qquad p(x) \to p(3) \cdot x + x^2 \int_{-1}^{1} p(x)\, dx.$$

Per ciascuna di essere determinare la forma canonica, ed una base in cui la matrice assume la forma canonica.

SOLUZIONI
BY GIMUSI 11.01.15

1. Determinare, per ciascuna delle seguenti matrici, la forma canonica reale e quella complessa (se diversa da quella reale). Determinare anche, nel caso reale ed eventualmente nel caso complesso, una possibile matrice di cambio di base che realizza il passaggio alla forma canonica.

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & -5 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 6 & 0 & 1 \\ 2 & 5 & 2 \\ -1 & 0 & 4 \end{pmatrix}$ (g) $\begin{pmatrix} 2 & 0 & 0 \\ -6 & 5 & 6 \\ 6 & -3 & -4 \end{pmatrix}$ (h) $\begin{pmatrix} -1 & -2 & -1 \\ 2 & 3 & 1 \\ 4 & 2 & 4 \end{pmatrix}$

(i) $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ (l) $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix}$ (m) $\begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ 3 & 3 & 3 \end{pmatrix}$ (n) $\begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ -3 & -3 & -3 \end{pmatrix}$

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $DET(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$ $\begin{cases} (1-\lambda)^3 + 2 - 3(1-\lambda) = \cancel{1} - \lambda^3 - \cancel{3\lambda} + 3\lambda^2 + \\ \cancel{1} + \cancel{2} - \cancel{3} + \cancel{3\lambda} = -\lambda^3 + 3\lambda^2 = \lambda^2(3-\lambda) = 0 \\ \rightsquigarrow \lambda_1 = \lambda_2 = 0 \ (MA=2) \quad \lambda_3 = 3 \ (MA=1) \end{cases}$

$\lambda_1 = \lambda_2 = 0 \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} X = 0$ $(MG=2)$ $X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\rightsquigarrow$ MG = MA = 2 DIAGONALIZZ. IN $\mathbb{R}$

$\lambda_3 = 3 \rightsquigarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} X = 0$ $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $M = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$ $\begin{cases} -\lambda^3 + 2\lambda = \lambda(2-\lambda^2) = 0 \\ \lambda_1 = 0 \quad \lambda_{2,3} = \pm\sqrt{2} \\ \rightsquigarrow \text{DIAGONALIZZ. IN } \mathbb{R} \end{cases}$ $\lambda_1 = 0 \rightsquigarrow X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda_{2,3} = \pm\sqrt{2} \rightsquigarrow \begin{pmatrix} \mp\sqrt{2} & 1 & 0 \\ 1 & \mp\sqrt{2} & 1 \\ 0 & 1 & \mp\sqrt{2} \end{pmatrix}$ $X_{2,3} = \begin{pmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{pmatrix}$ $\rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$ $M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$ $\begin{cases} -\lambda^3 - \lambda + \lambda = 0 \\ \lambda_1 = \lambda_2 = \lambda_3 = 0 \end{cases}$ MA = 3 RANGO = 2 MG = 1 $\rightsquigarrow X_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $AM = MJ \rightsquigarrow \begin{cases} Ax_2 = x_1 \\ Ax_3 = x_2 \end{cases}$ $\rightsquigarrow x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

JORDANIZZ. 1 BLOCCO

(d) $\begin{pmatrix} 0 & -5 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix} -\lambda & -5 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \begin{cases} -\lambda^3+\lambda-5\lambda = -\lambda^3-4\lambda = \\ = -\lambda(\lambda^2+4) = 0 \\ \lambda_1 = 0 \quad \lambda_{2,3} = \pm 2i \end{cases} \rightsquigarrow$

DIAG. IN $\mathbb{C}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix} J_{\mathbb{C}}$$

FORMA DI J-REALE $\rightsquigarrow$ $J_{\mathbb{R}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$ $\quad \lambda_1 = 0 \rightsquigarrow X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\quad \lambda_2 = 2i \rightsquigarrow \begin{pmatrix} -2i & -5 & 0 \\ 1 & -2i & 1 \\ 0 & 1 & -2i \end{pmatrix} X = 0$

$$\rightsquigarrow \begin{pmatrix} -2i & -5 & 0 \\ 2i & 5 & 2i \\ 0 & 1 & -2i \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2i & -5 & 0 \\ 0 & -1 & 2i \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow X_2 = \begin{pmatrix} 5i \\ 2 \\ -i \end{pmatrix} \qquad X_3 = \overline{X_2} = \begin{pmatrix} -5i \\ 2 \\ i \end{pmatrix}$$

$M_{\mathbb{C}} = \begin{pmatrix} 1 & 5i & -5i \\ 0 & 2 & 2 \\ -1 & -i & i \end{pmatrix}$ $\quad M_{\mathbb{R}} = ? \quad AM = MJ_{\mathbb{R}} \rightsquigarrow \begin{cases} AV_1 = 0 \rightsquigarrow V_1 = X_1 \\ AV_2 = -2V_3 \\ AV_3 = 2V_2 \end{cases}$ VD. LEZ. 58

$$\begin{cases} Av = (2i)v \\ A\bar{v} = (-2i)\bar{v} \end{cases} \quad v = V_2 + iV_3 \rightsquigarrow Av = (2i)v = 2i V_2 - 2V_3 = -2V_3 + i(2V_2)$$

$$v = V_2 + iV_3 = \begin{pmatrix} 5i \\ 2 \\ -i \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} \rightsquigarrow V_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \quad V_3 = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} \quad M_{\mathbb{R}} = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$\underbrace{\frac{1}{8}\begin{pmatrix} -2 & 0 & -10 \\ 0 & 5 & 0 \\ 2 & 0 & 2 \end{pmatrix}}_{M_{\mathbb{R}}^{-1}} \underbrace{\begin{pmatrix} 0 & -5 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix}}_{M_{\mathbb{R}}} = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 5 \\ 0 & -8 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix} = J_{\mathbb{R}}$$

(e) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & 2 \\ 2 & 0 & 1-\lambda \end{vmatrix} = 0 \begin{cases} (1-\lambda)^3+8 = [(1-\lambda)+2][(1-\lambda)^2-2(1-\lambda)+4] \\ = (3-\lambda)(1+\lambda^2-\cancel{2\lambda}-2+\cancel{2\lambda}+4) = (3-\lambda)(\lambda^2+3) = 0 \\ \rightsquigarrow \lambda_1 = 3 \quad \lambda_{2,3} = \pm\sqrt{3}\,i \end{cases}$ DIAG. IN $\mathbb{C}$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & i\sqrt{3} & 0 \\ 0 & 0 & -i\sqrt{3} \end{pmatrix} \rightsquigarrow J_{\mathbb{R}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & -\sqrt{3} & 0 \end{pmatrix} \qquad \lambda_1 = 3 \rightsquigarrow \begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 2 \\ 2 & 0 & -2 \end{pmatrix} X = 0 \quad X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = i\sqrt{3} \rightsquigarrow \begin{pmatrix} 1-i\sqrt{3} & 2 & 0 \\ 0 & 1-i\sqrt{3} & 2 \\ 2 & 0 & 1-i\sqrt{3} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1-i\sqrt{3} & 2 & 0 \\ 0 & 1-i\sqrt{3} & 2 \\ 0 & -4 & (1-i\sqrt{3})^2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1-i\sqrt{3} & 2 & 0 \\ 0 & 1-i\sqrt{3} & 2 \\ 0 & 0 & \overset{0}{\cancel{(1-i\sqrt{3})^3+8}} \end{pmatrix} X = 0$$

$$x_2=\begin{pmatrix}1+i\sqrt{3}\\-2\\1-i\sqrt{3}\end{pmatrix}\quad X_3=\bar{X}_2=\begin{pmatrix}1-i\sqrt{3}\\-2\\1+i\sqrt{3}\end{pmatrix}\quad M_{\mathbb{C}}=\begin{pmatrix}1&1+i\sqrt{3}&1-i\sqrt{3}\\1&-2&-2\\1&1-i\sqrt{3}&1+i\sqrt{3}\end{pmatrix}\quad M_{\mathbb{R}}=\begin{pmatrix}1&1&\sqrt{3}\\1&-2&0\\1&1&-\sqrt{3}\end{pmatrix}$$

(f) $\begin{pmatrix}6&0&1\\2&5&2\\-1&0&5\end{pmatrix}$ $|A-\lambda I|=\begin{vmatrix}6-\lambda&0&1\\2&5-\lambda&2\\-1&0&5-\lambda\end{vmatrix}=\begin{cases}(6-\lambda)(5-\lambda)(5-\lambda)+(5-\lambda)=\\(5-\lambda)(\lambda^2-10\lambda+25)=(5-\lambda)^3=0\\\lambda_1=\lambda_2=\lambda_3=5\end{cases}$

$\rightsquigarrow\begin{pmatrix}1&0&1\\2&0&2\\-1&0&-1\end{pmatrix}X=0$ MG=2<MA=3 JORDANIZZ. 2 BLOCCHI $\rightsquigarrow\begin{pmatrix}5&0&0\\0&5&1\\0&0&5\end{pmatrix}$ $\begin{cases}Ax_1=5x_1\\Ax_2=5x_2\\Ax_3=x_2+5x_3\end{cases}$

$X_1=\begin{pmatrix}1\\0\\-1\end{pmatrix}\quad X_2=\begin{pmatrix}1\\2\\-1\end{pmatrix}\quad X_3=\begin{pmatrix}1\\0\\0\end{pmatrix}\rightsquigarrow M=\begin{pmatrix}1&1&1\\0&2&0\\-1&-1&0\end{pmatrix}$

(g) $\begin{pmatrix}2&0&0\\-6&5&6\\6&-3&-5\end{pmatrix}$ $|A-\lambda I|=\begin{vmatrix}2-\lambda&0&0\\-6&5-\lambda&6\\6&-3&-5-\lambda\end{vmatrix}=\begin{cases}(2-\lambda)(5-\lambda)(-5-\lambda)+18(2-\lambda)=\\(2-\lambda)(\lambda^2-\lambda-2)=-(\lambda-2)^2(\lambda+1)=0\\\lambda_1=-1\quad\lambda_{2,3}=2\end{cases}$ MG=MA=2

$\lambda_1=-1\rightsquigarrow\begin{pmatrix}3&0&0\\-6&6&6\\6&-3&-3\end{pmatrix}X=0\quad X_1=\begin{pmatrix}0\\1\\-1\end{pmatrix}\quad\lambda_{2,3}=2\rightsquigarrow\begin{pmatrix}0&0&0\\-6&3&6\\6&-3&-6\end{pmatrix}\quad X_2=\begin{pmatrix}1\\0\\1\end{pmatrix}\quad X_3=\begin{pmatrix}1\\2\\0\end{pmatrix}$

DIAGONALIZ. IN $\mathbb{R}$ $\rightsquigarrow\begin{pmatrix}-1&0&0\\0&2&0\\0&0&2\end{pmatrix}\rightsquigarrow M_{\mathbb{R}}=\begin{pmatrix}0&1&1\\1&0&2\\-1&1&0\end{pmatrix}$

(h) $\begin{pmatrix}-1&-2&-1\\2&3&1\\5&2&5\end{pmatrix}$ $|A-\lambda I|=\begin{vmatrix}-1-\lambda&-2&-1\\2&3-\lambda&1\\5&2&5-\lambda\end{vmatrix}=\begin{cases}(\lambda^2-2\lambda-3)(5-\lambda)-8-5+5(3-\lambda)-2(-1-\lambda)+\\+5(5-\lambda)=5\lambda^2-8\lambda-12-\lambda^3+2\lambda^2+3\lambda-\cancel{12}+\cancel{12}+\\-5\lambda+2+2\lambda+16-5\lambda=-\lambda^3+6\lambda^2-11\lambda+6=0\end{cases}$

	-1	6	-11	6
1		-1	5	-6
	-1	5	-6	0

$\rightsquigarrow(\lambda-1)(-\lambda^2+5\lambda-6)=$
$=-(\lambda-1)(\lambda-2)(\lambda-3)=0$
$\lambda_1=1\quad\lambda_2=2\quad\lambda_3=3$ $\rightsquigarrow\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}$

$\lambda_1=1\rightsquigarrow\begin{pmatrix}-2&-2&-1\\2&2&1\\5&2&3\end{pmatrix}X=0\quad X_1=\begin{pmatrix}2\\-1\\-2\end{pmatrix}\quad\lambda_2=2\rightsquigarrow\begin{pmatrix}-3&-2&-1\\2&1&1\\5&2&2\end{pmatrix}X=0\quad X_2=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$

$$\lambda_3=3 \rightsquigarrow \begin{pmatrix} -5 & -2 & -1 \\ 2 & 0 & 1 \\ 5 & 2 & 1 \end{pmatrix} X=0 \quad x_3=\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \rightsquigarrow M_{\mathbb{R}}=\begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & -1 \\ -2 & -1 & -2 \end{pmatrix}$$

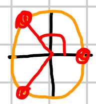
(i) $$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \quad |A-\lambda I|=\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = \begin{cases} (1-\lambda)^3+8+8-12(1-\lambda)=1-\lambda^3-3\lambda+ \\ +3\lambda^2+16-12+12\lambda=-\lambda^3+3\lambda^2+9\lambda+5=0 \end{cases}$$

$$\begin{array}{r|rrr|r} & -1 & 3 & 9 & 5 \\ -1 & & 1 & -5 & -5 \\ \hline & -2 & 5 & 5 & // \end{array} \rightsquigarrow \begin{cases} (\lambda+2)(-\lambda^2+5\lambda+5) \\ =(\lambda+2)^2(\lambda-5)=0 \\ \lambda_{1,2}=-2 \quad \lambda_3=5 \end{cases}$$

$$\lambda_{1,2}=-1 \rightsquigarrow \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} X=0 \quad x_1=\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad x_2=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_3=5 \rightsquigarrow \begin{pmatrix} -5 & 2 & 2 \\ 2 & -5 & 2 \\ 2 & 2 & -5 \end{pmatrix} X=0 \quad X_3=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad M_{\mathbb{R}}=\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

(ℓ) $$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix} \quad |A-\lambda I|=\begin{vmatrix} -\lambda & 2 & 0 \\ 0 & -\lambda & 2 \\ 2 & 0 & -\lambda \end{vmatrix} = \begin{cases} -\lambda^3+8=0 \quad \lambda^3=8 \quad \lambda_1=2 \\ \lambda_{2,3}=-1\pm i\sqrt{3} \end{cases}$$



$$\lambda_1=2 \rightsquigarrow \begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 2 \\ 2 & 0 & -2 \end{pmatrix} X=0 \quad X_1=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda_2=-1+i\sqrt{3} \rightsquigarrow \begin{pmatrix} 1-i\sqrt{3} & 2 & 0 \\ 0 & 1-i\sqrt{3} & 2 \\ 2 & 0 & 1-i\sqrt{3} \end{pmatrix} X=0$$

$$X_2=\begin{pmatrix} 1+i\sqrt{3} \\ -2 \\ 1-i\sqrt{3} \end{pmatrix} \quad X_2=\bar{X}_3=\begin{pmatrix} 1-i\sqrt{3} \\ -2 \\ 1+i\sqrt{3} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1+i\sqrt{3} & 0 \\ 0 & 0 & -1-i\sqrt{3} \end{pmatrix} \rightsquigarrow J_{\mathbb{R}}=\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & \sqrt{3} \\ 0 & -\sqrt{3} & -1 \end{pmatrix}$$

$$\rightsquigarrow M_{\mathbb{C}}=\begin{pmatrix} 1 & 1+i\sqrt{3} & 1-i\sqrt{3} \\ 1 & -2 & -2 \\ 1 & 1-i\sqrt{3} & 1+i\sqrt{3} \end{pmatrix} \rightsquigarrow M_{\mathbb{R}}=\begin{pmatrix} 1 & 1 & \sqrt{3} \\ 1 & -2 & 0 \\ 1 & 1 & -\sqrt{3} \end{pmatrix}$$

(m) $$\begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ 3 & 3 & 3 \end{pmatrix} \begin{cases} RANGO=1 \Rightarrow \lambda_1=\lambda_2=0 \\ Tr(A)=0 \Rightarrow \lambda_3=0 \end{cases} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{cases} Ax_1=0 \\ Ax_2=0 \quad x_2\in KER \\ Ax_3=x_2 \quad x_2\in IM \end{cases}$$

$$x_1=\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad X_2=\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \quad X_3=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow M=\begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

(m) $\begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ -3 & -3 & -3 \end{pmatrix} \begin{cases} \text{RANGO}=1 \Rightarrow \lambda_1=\lambda_2=0 \quad MA=MG=2 \\ Tr(A)=-6 \Rightarrow \lambda_3=-6 \quad MA=MG=1 \end{cases} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

$\lambda_{1,2}=0 \rightsquigarrow X_2=\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad X_2=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \lambda_3=-6 \rightsquigarrow \begin{pmatrix} 5 & -1 & -1 \\ -2 & 5 & -2 \\ -3 & -3 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 5 & -1 & -1 \\ 0 & 18 & -12 \\ 0 & -18 & 12 \end{pmatrix} \rightsquigarrow$

$\rightsquigarrow \begin{pmatrix} 5 & -1 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix} X=0 \quad X_3=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightsquigarrow M=\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix}$

2. Determinare per quali valori del parametro a la matrice $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & a \\ 0 & 0 & 1 \end{pmatrix}$ è diagonalizzabile.

$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & a \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} \lambda_1=2 \\ \lambda_2=\lambda_3=1 \rightsquigarrow \end{matrix} \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & a \\ 0 & 0 & 0 \end{pmatrix}$ DIAGON. $\Leftrightarrow MG=MA=2 \Rightarrow a=3$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\lambda_1=2 \rightsquigarrow \begin{pmatrix} -1 & 1 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{pmatrix} X_1=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \lambda_{2,3}=1 \rightsquigarrow X_2=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad X_3=\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \rightsquigarrow M=\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & -1 \end{pmatrix}$

3. Determinare per quale valore del parametro reale a le due matrici

$$A=\begin{pmatrix} 1 & 7 & 5 \\ 0 & 2 & -2 \\ 0 & 0 & a \end{pmatrix} \qquad B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & a \end{pmatrix}$$

sono simili.

$\begin{cases} Tr(A)=Tr(B)=3+a \\ DET(A)=DET(B)=2a \end{cases} \rightsquigarrow \begin{cases} B \text{ È SIMILE AD } A \Leftrightarrow A \text{ È DIAGONALIZZABILE} \\ a\neq 1,\ a\neq 2 \Rightarrow A \text{ È DIAGONALIZZ. } (3\ \lambda \text{ DISTINTI}) \end{cases}$

$\lambda=a=1 \rightsquigarrow \begin{pmatrix} 0 & 7 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda=a=2 \rightsquigarrow \begin{pmatrix} -1 & 7 & 5 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{pmatrix}$ $MG=1<MA=2$ $\rightsquigarrow A$ NON DIAG.

$\rightsquigarrow A$ E B SIMILI $\forall a \in \mathbb{R}-\{1,2\}$

4. Consideriamo le seguenti applicazioni lineari $f : \mathbb{R}^3 \to \mathbb{R}^3$:

$$(a)\ (x, y, z) \to (y, z, z), \quad (b)\ (x, y, z) \to (x, x + y, x + y + z).$$

Per ciascuna di essere determinare la forma canonica, ed una base in cui la matrice assume la forma canonica.

(a) $(x,y,z) \rightsquigarrow (y,z,z)$

$$\overset{A}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ z \end{pmatrix} \quad \begin{cases} \lambda_{1,2} = 0 & MG=1<MA=2 \\ \lambda_3 = 1 & \text{JORDANIZZ.} \end{cases}$$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{cases} Ax_1 = 0 & x_1 \in KER(A) \\ Ax_2 = x_1 & x_1 \in IM(A) \\ Ax_3 = x_3 \end{cases} \quad x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(A-I)x_3 = 0 \rightsquigarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} X = 0 \quad x_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightsquigarrow M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) $(x,y,z) \rightsquigarrow (x, x+y, x+y+z)$

$$\overset{A}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \lambda_{1,2,3} = 1 \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \rightsquigarrow$$

MG=1 < MA=3
JORDANIZZ.
1 BLOCCO

$$\rightsquigarrow C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{cases} Ax_1 = x_1 \\ Ax_2 = x_1 + x_2 \\ Ax_3 = x_2 + x_3 \end{cases} \quad \begin{cases} (A-I)x_1 = 0 \\ (A-I)x_2 = x_1 \\ (A-I)x_3 = x_2 \end{cases}$$

$$x_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in KER, IM\ (A-I) \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \rightsquigarrow M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

5. Consideriamo le seguenti applicazioni lineari $f : \mathbb{R}_{\le 2}[x] \to \mathbb{R}_{\le 2}[x]$:

(a) $p(x) \to p(-x)$, (b) $p(x) \to p(x) + p(-x)$, (c) $p(x) \to p(x) + p(2x)$,
(d) $p(x) \to p'(x)$, (e) $p(x) \to p(x) + p(x+1)$, (f) $p(x) \to p(7) \cdot x^2$,
(g) $p(x) \to xp'(x) + 2p(x)$, (h) $p(x) \to (x+2)p'(x)$, (i) $p(x) \to p(3) \cdot x + x^2 \int_{-1}^{1} p(x)\,dx$.

Per ciascuna di essere determinare la forma canonica, ed una base in cui la matrice assume la forma canonica.

(a) $p(x) = ax^2 + bx + c \rightsquigarrow ax^2 - bx + c = p(-x)$ $\quad A = C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad M = I$

(b) $p(x) = ax^2 + bx + c \rightsquigarrow 2ax^2 + 2c = p(x) + p(-x)$ $\quad A = C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad M = I$

(c) $p(x) = ax^2 + bx + c \rightsquigarrow 5ax^2 + 3bx + 2c = p(x) + p(2x)$ $\quad A = C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad M = I$

(d) $p(x) = ax^2 + bx + c = 2ax + b = p'(x)$ $\quad A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \lambda_{1,2,3} = 0$

MG = 1 < MA = 3
JORDANIZZ.
1 BLOCCO

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{cases} Ax_1 = 0 & x_1 \in \ker \\ Ax_2 = x_1 & x_1 \in \mathrm{Im} \\ Ax_3 = x_2 \end{cases}$ $\quad x_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$

$M = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

(e) $p(x) = ax^2 + bx + c = ax^2 + bx + c + a(x^2 + 2x + 1) + b(x+1) + c =$

$= 2ax^2 + (2a + 2b)x + (a + b + 2c) = p(x) + p(x+1)$

$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \quad \lambda_{1,2,3} = 2 \quad \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} X = 0$

MG = 1 < MA = 3
JORDANIZZ.
1 BLOCCO

$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$\begin{cases} Ax_1 = 2x_1 \\ Ax_2 = x_1 + 2x_2 \\ Ax_3 = x_2 + 2x_3 \end{cases}$ $\quad$ $x_1 \in \ker, \mathrm{Im}$ $\begin{cases} (A - 2I)x_1 = 0 \\ (A - 2I)x_2 = x_1 \\ (A - 2I)x_3 = x_2 \end{cases}$ $\quad x_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \quad M = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 1 & 0 & 0 \end{pmatrix}$

(f) $p(x) = ax^2+bx+c = (49a+7b+c)x^2 = p(7)\cdot x^2$

$$A = \begin{pmatrix} 49 & 7 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{cases} \lambda_1 = 49 \\ \lambda_{2,3} = 0 \end{cases} \rightsquigarrow \begin{pmatrix} 49 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ 0 \\ -49 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ -7 \\ 0 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -7 \\ 0 & -49 & 0 \end{pmatrix}$$

MG = MA = 2

(g) $p(x) = ax^2+bx+c \rightsquigarrow x(2ax+b)+2ax^2+2bx+2c = 5ax^2+3bx+2c = xp'(x)+2p(x)$

$$A = C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad M = I$$

(h) $p(x) = ax^2+bx+c \rightsquigarrow (x+2)(2ax+b) = 2ax^2+bx+4ax+2b = 2ax^2+(4a+b)x+2b = (x+2)p'(x)$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \quad \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 1 \\ \lambda_3 = 0 \end{cases} \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_1 = 2 \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 4 & -1 & 0 \\ 0 & 2 & -2 \end{pmatrix} x = 0 \quad x_1 = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

$$\lambda_2 = 1 \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} x = 0 \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \lambda_3 = 0 \rightsquigarrow x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow M = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

(i) $p(x) = ax^2+bx+c \rightsquigarrow (9a+3b+c)x + x^2 \left[a\frac{x^3}{3} + b\frac{x^2}{2} + cx + d \right]_{-1}^{1} =$

$= (9a+3b+c)x + x^2(a/3 + \cancel{b/2} + c + a/3 - \cancel{b/2} + c) = (2a/3+2c)x^2 + (9a+3b+c)x$

$$A = \begin{pmatrix} 2/3 & 0 & 2 \\ 9 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad |A-\lambda I| = \begin{vmatrix} 2/3-\lambda & 0 & 0 \\ 9 & 3-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = \left(\tfrac{2}{3}-\lambda\right)(3-\lambda)(-\lambda) = 0 \quad \begin{cases} \lambda_1 = 2/3 \\ \lambda_2 = 3 \\ \lambda_3 = 0 \end{cases}$$

$$\rightsquigarrow \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_1 = 2/3 \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 9 & 7/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix} x = 0 \quad x_1 = \begin{pmatrix} 7 \\ -27 \\ 0 \end{pmatrix} \quad \lambda_2 = 3 \rightsquigarrow \begin{pmatrix} -7/3 & 0 & 0 \\ 9 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} x = 0$$

$$x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \lambda_3 = 0 \rightsquigarrow x_3 = \begin{pmatrix} 3 \\ -26/3 \\ -1 \end{pmatrix} \rightsquigarrow M = \begin{pmatrix} 7 & 0 & 3 \\ -27 & 1 & -26/3 \\ 0 & 0 & -1 \end{pmatrix}$$