Forme canoniche 1

Argomenti: matrici simili **Difficoltà:** ⋆⋆⋆

Prerequisiti: autovalori, autovettori, forme canoniche, matrici di cambio di base

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1) Consideriamo l'applicazione lineare $f : \mathbb{R}^4 \to \mathbb{R}^3$ definita da

$$f(x, y, z, w) = (x + y + z, y + z + w, w - x).$$

Determinare quali delle seguenti matrici rappresentano f per un'opportuna scelta delle basi in partenza ed arrivo (nei casi affermativi non sarebbe male anche fornire un esempio di tali basi).

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 5 \end{pmatrix}$ (g) $\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & \sqrt[5]{2} & \sqrt[5]{2} & 0 \\ 5 & 0 & 0 & 5 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \end{pmatrix}$

2) In ciascuna delle righe seguenti vengono presentate k matrici. Di queste, $(k-1)$ sono simili tra loro. Si richiede di determinare l'intrusa. Inoltre, per ogni coppia di matrici simili, non sarebbe male determinare la matrice di cambio di base (o meglio, una delle possibili matrici di cambio di base) che realizza la similitudine.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ \sqrt{3} & 2 \end{pmatrix}$ $\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$ $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$ $\begin{pmatrix} 10 & 18 \\ -4 & -7 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}$ $\begin{pmatrix} 5 & 5 \\ -2 & -1 \end{pmatrix}$ $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ $\begin{pmatrix} 4 & -5 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 6 & -4 \\ 4 & -2 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 7 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 6 & -2 & 0 \\ 5 & 4 & 3 \end{pmatrix}$ $\begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(e) $\begin{pmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ -3 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 7 & 5 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ $\begin{pmatrix} 3 & 3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix}$ $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ $\begin{pmatrix} 3 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

SOLUZIONI REV.01
BY GIMUSI 07/01/15

1) Consideriamo l'applicazione lineare $f : \mathbb{R}^4 \to \mathbb{R}^3$ definita da

$$f(x, y, z, w) = (x + y + z, y + z + w, w - x).$$

Determinare quali delle seguenti matrici rappresentano f per un'opportuna scelta delle basi in partenza ed arrivo (nei casi affermativi non sarebbe male anche fornire un esempio di tali basi).

(a) RANK=3 $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \end{pmatrix}$ (d) RANK=1 $\begin{pmatrix} 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ (f) RANK=1 $\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 5 \end{pmatrix}$ (g) $\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & \sqrt[5]{2} & \sqrt[5]{2} & 0 \\ 5 & 0 & 0 & 5 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \end{pmatrix}$

$f(x,y,z,w) = (x+y+z, y+z+w, w-x)$

$$\begin{cases} e_1: (1,0,0,0) \rightsquigarrow (1,0,-1) = f(e_1) \\ e_2: (0,1,0,0) \rightsquigarrow (1,1,0) = f(e_2) \\ e_3: (0,0,1,0) \rightsquigarrow (1,1,0) = f(e_3) \\ e_4: (0,0,0,1) \rightsquigarrow (0,1,1) = f(e_4) \end{cases} \rightsquigarrow A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow$$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} RANK(A) = 2 \\ KER(f) = n-2 = 2 \end{cases}$$

$\Rightarrow$ NON RAPPRESENTANO f LE MATRICI (a) (d) (f)

METODO 1 - BOVINO

(b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

B' DI $\mathbb{R}^4$		BASE IM	B'' DI $\mathbb{R}^3$
1) e_1	$\rightsquigarrow$	$e_1 - e_3$	a) $e_1 - e_3$
2) e_2	$\rightsquigarrow$	$e_1 + e_2$	b) $(e_1 + e_2)/2$
3) $e_2 - e_3$	$\rightsquigarrow$	0	c) e_1
4) $e_1 - e_2 + e_4$	$\rightsquigarrow$	0	

BASE KER $\rightarrow$ [3), 4)]

(c) $\begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \end{pmatrix}$

B' DI $\mathbb{R}^4$			B'' DI $\mathbb{R}^3$
1) $e_2 - e_3$	$\rightsquigarrow$	0	a) $(e_1 + e_2)/3$
2) e_1	$\rightsquigarrow$	$e_1 - e_3$	b) e_1
3) e_2	$\rightsquigarrow$	$e_1 + e_2$	c) $(e_1 - e_3)/7$
4) $e_1 - e_2 + e_4$	$\rightsquigarrow$	0	

(e)

C_1	C_2	C_3	C_4
1	2	3	4
5	6	7	8
9	10	11	12

B' DI $\mathbb{R}^4$

1) $e_1 \rightsquigarrow e_1 - e_3$
2) $e_2 \rightsquigarrow e_1 + e_2$
3) $3e_2 - e_1 - e_3 \rightsquigarrow e_1 + 2e_2 + e_3$
4) $2e_2 - e_1 + e_4 \rightsquigarrow e_1 + 3e_2 + 2e_3$

B'' DI $\mathbb{R}^3$

a) $-\frac{e_1}{4} + \frac{5}{4}e_2 + \frac{5}{2}e_3$
b) $\frac{e_1}{4} - \frac{e_2}{4} - \frac{5}{2}e_3$
c) e_3

FISSIAMO e_1 E e_2, 1° E 2° VETTORE BASE B' IN $\mathbb{R}^4$

$\{a, b, c\} \equiv$ BASE IN ARRIVO $\rightsquigarrow \begin{cases} a + 5b + 9c = e_1 - e_3 & (C_1) \\ 2a + 6b + 10c = e_1 + e_2 & (C_2) \end{cases}$

PONIAMO $\rightsquigarrow c = e_3 \begin{cases} a + 5b = e_1 - 10e_3 \\ 2a + 6b = e_1 + e_2 - 10e_3 \end{cases} \rightsquigarrow \begin{cases} a = -\frac{e_1}{4} + \frac{5}{4}e_2 + \frac{5}{2}e_3 \\ b = \frac{e_1}{4} - \frac{e_2}{4} - \frac{5}{2}e_3 \end{cases}$

$C_3 = 2C_2 - C_1 \rightsquigarrow$ 3° IN B': $2e_2 - e_1 + \underbrace{(e_2 - e_3)}_{\in \text{KER}} = 3e_2 - e_1 - e_3$

$f(3e_2 - e_1 - e_3) = f(2e_2 - e_1) = 2(e_1 + e_2) - (e_1 - e_3) = e_1 + 2e_2 + e_3$

$C_4 = 2C_3 - C_2 = 4C_2 - 2C_1 - C_2 = 3C_2 - 2C_1 \rightsquigarrow$ 4° IN B':

$3e_2 - 2e_1 + \underbrace{(e_1 - e_2 + e_4)}_{\in \text{KER}} = 2e_2 - e_1 + e_4$

$f(2e_2 - e_1 + e_4) = f(3e_2 - 2e_1) = 3(e_1 + e_2) - 2(e_1 - e_3) = e_1 + 3e_2 + 2e_3$

(g)

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & \sqrt[5]{2} & \sqrt[5]{2} & 0 \\ 5 & 0 & 0 & 5 \end{pmatrix}$$

B' DI $\mathbb{R}^4$

1) $e_1 \rightsquigarrow e_1 - e_3$
2) $e_2 \rightsquigarrow e_1 + e_2$
3) $2e_2 - e_3 \rightsquigarrow e_1 + e_2$
4) $2e_1 - e_2 + e_4 \rightsquigarrow e_1 - e_3$

B'' DI $\mathbb{R}^3$

a) $e_1/2$
b) $(e_1 + e_2)/\sqrt[5]{2}$
c) $-e_3/5$

(h)

$$\begin{pmatrix} 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \end{pmatrix}$$

B' DI $\mathbb{R}^4$

1) $e_1 \rightsquigarrow e_1 - e_3$
2) $e_2 \rightsquigarrow e_1 + e_2$
3) $e_1 + e_2 - e_3 \rightsquigarrow e_1 - e_3$
4) $e_1 + e_4 \rightsquigarrow e_1 + e_2$

B'' DI $\mathbb{R}^3$

a) $\frac{e_1}{4} + \frac{3}{8}e_2 + \frac{3}{8}e_3$
b) $\frac{e_1}{4} - \frac{e_2}{8} - \frac{e_3}{8}$
c) $-e_3$

FISSIAMO e_1 E e_2, 1° E 2° VETTORE BASE B' IN $\mathbb{R}^4$

$\{a, b, c\} \equiv$ BASE IN ARRIVO $\rightsquigarrow \begin{cases} a + 3b + c = e_1 - e_3 & (C_1) \\ 3a + b + 3c = e_1 + e_2 & (C_2) \end{cases}$

PONIAMO $\rightsquigarrow c = -e_3 \begin{cases} a + 3b = e_1 \to a = e_1 - 3b \\ 3e_1 - 8b = e_1 + e_2 + e_3 \end{cases} \begin{cases} a = e_1 - \frac{3}{4}e_1 + \frac{3}{8}e_2 + \frac{3}{8}e_3 = \frac{e_1}{4} + \frac{3}{8}e_2 + \frac{3}{8}e_3 \\ b = \frac{e_1}{4} - \frac{e_2}{8} - \frac{e_3}{8} \end{cases}$

METODO 2 (Vd. LEZ. 37) - SEMIBOVINO

$$\begin{cases} f(e_1) = e_1 - e_3 \\ f(e_2) = e_1 + e_2 \\ f(e_3) = e_1 + e_2 \\ f(e_4) = e_2 + e_3 \end{cases} \rightsquigarrow \begin{cases} f(e_1) = e_1 - e_3 \\ f(e_2) = e_1 + e_2 \\ f(e_2 - e_3) = 0 \\ f(e_1 - e_2 + e_4) = 0 \end{cases}$$

BASI DELLA FORMA CANONICA

B'_C DI $\mathbb{R}^4$	B''_C DI $\mathbb{R}^3$
$v_1 = e_1$	$w_1 = e_1 - e_3$
$v_2 = e_2$	$w_2 = e_1 + e_2$
$v_3 = e_2 - e_3$	$w_3 = e_3$
$v_4 = e_1 - e_2 - e_4$	

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \boxed{C v_\varepsilon = w_\varepsilon}$$

$$\begin{cases} v_\varepsilon \rightsquigarrow \text{IN BASE } B'_C \\ w_\varepsilon \rightsquigarrow \text{IN BASE } B''_C \end{cases}$$

(a) $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

B' DI $\mathbb{R}^4$		B'' DI $\mathbb{R}^3$	
1)	v_1	a)	w_1
2)	v_2	b)	$w_2/2$
3)	v_3	c)	w_3
4)	v_4		

(c) $$\begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \end{pmatrix}$$

B' DI $\mathbb{R}^4$		B'' DI $\mathbb{R}^3$	
1)	v_3	a)	$w_1/3$
2)	v_2	b)	w_3
3)	v_1	c)	$w_2/7$
4)	v_4		

(e) $$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

B' DI $\mathbb{R}^4$		B'' DI $\mathbb{R}^3$	
1)	$v_1 + 5v_2$	a)	$w_1 - 9w_3$
2)	$2v_1 + 6v_2$	b)	$w_2 + \frac{5}{3} w_3$
3)	$3v_1 + 7v_2 + v_3$	c)	w_3
4)	$4v_1 + 8v_2 + v_4$		

c_1: $w_1 - 9w_3 + 5w_2 + 9w_3 = w_1 + 5w_2 = f(v_1 + 5v_2)$

c_2: $2w_1 - 18w_3 + 6w_2 + 10w_3 = 2w_1 + 6w_2 - 8w_3 = f(2v_1 + 6v_2) - 8w_3$

$c_3 = 2c_2 - c_1 \rightsquigarrow 3v_1 + 7v_2$ $\quad$ $c_4 = 3c_2 - 2c_1 \rightsquigarrow 4v_1 + 8v_2$

C_1

(g) $\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & \sqrt[5]{2} & \sqrt[5]{2} & 0 \\ 5 & 0 & 0 & 5 \end{pmatrix}$

$C_1: 2w_1 - 10w_3 + 10w_3 = 2w_1 = f(2v_1)$

$\mathcal{B}'$ DI $\mathbb{R}^4$
1) $2v_1$
2) $\sqrt[5]{2}\, v_2$
3) $\sqrt[5]{2}\, v_2 + v_3$
4) $2v_1 + v_4$

$\mathcal{B}''$ DI $\mathbb{R}$
a) $w_1 - 5w_3$
b) w_2
c) $2w_3$

$C_1 \quad C_2$

(4) $\begin{pmatrix} 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \end{pmatrix}$

$C_2: 3w_1 - 3w_3 + w_2 + 3w_3 = 3w_1 + w_2 = f(3v_1 + v_2)$

$\mathcal{B}'$ DI $\mathbb{R}^4$
1) $v_1 + 3v_2$
2) $3v_1 + v_2$
3) $v_1 + 3v_2 + v_3$
4) $v_1 + 3v_2 + v_4$

$\mathcal{B}''$ DI $\mathbb{R}$
a) $w_1 - w_3$
b) w_2
c) w_3

METODO 3 – MATRICIALE (TEOREMA LEZ. 37)

$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ ≡ MATRICE ASSOCIATA A f NELLE BASI $\mathcal{e}$

IN PARTENZA: e_1, e_2, e_3, e_4 IN $\mathbb{R}^4$
IN ARRIVO e_1, e_2, e_3 IN $\mathbb{R}^3$

$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ ≡ RANGO = 2

DIM(KER f) = 4 − 2 = 2
DIM(IM f) = 2

$\Rightarrow \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ ≡ FORMA CANONICA DI f

IN QUALI BASI?

1) BASE DEL KER: $v_3 = (0, -1, 1, 0) \quad v_4 = (1, -1, 0, 1)$

2) COMPLETAMENTO BASE KER: $v_1 = e_1 = (1,0,0,0) \quad v_2 = e_2 = (0,1,0,0)$

3) BASE DI IM: $w_1 = f(v_1) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad w_2 = f(v_2) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad w_3 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ BASI $\mathcal{E}$ DELLA FORMA CANONICA
IN PARTENZA: $v_1, v_2, v_3, v_4 \in \mathbb{R}^4$
IN ARRIVO: $w_1, w_2, w_3 \in \mathbb{R}^3$

IN TERMINI MATRICIALI:

NELLE BASI e : $A x_e = y_e$ $\qquad x_e \in \mathbb{R}^4 \qquad y_e = f(x_e) \in \mathbb{R}^3$

NELLE BASI $\mathcal{E}$: $C x_\mathcal{E} = y_\mathcal{E}$ $\qquad x_\mathcal{E} \in \mathbb{R}^4 \qquad y_\mathcal{E} = f(x_\mathcal{E}) \in \mathbb{R}^3$

CAMBIO DI BASI: $y_e = M_A y_\mathcal{E} \qquad x_e = N_A x_\mathcal{E} \qquad y_\mathcal{E} \in \mathbb{R}^3 \qquad x_\mathcal{E} \in \mathbb{R}^4$

BASE IM: w_1, w_2 $\qquad$ BASE KER: v_3, v_4

$$M_A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}_{3\times 3} \quad (\text{colonne } w_1, w_2, w_3) \qquad N_A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times 4} \quad (\text{colonne } v_1, v_2, v_3, v_4)$$

$$\Rightarrow A x_e = y_e \rightsquigarrow A N_A x_\mathcal{E} = M_A y_\mathcal{E} \rightsquigarrow \left(M_A^{-1} A N_A\right) x_\mathcal{E} = y_\mathcal{E}$$

$$\Rightarrow C = M_A^{-1} A N_A$$

SIA B UN'ALTRA MATRICE ASSOCIATA A f:

RANGO = 2

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

IN QUALI BASI? $\rightsquigarrow$ CERCHIAMO LE BASI DELLA FORMA CANONICA

1) BASE DEL KER: $\hat{v}_3 = (1, -2, 1, 0) \qquad \hat{v}_4 = (2, -3, 0, 1)$

2) COMPLETAMENTO BASE KER: $\hat{v}_1 = e_1 = (1,0,0,0) \qquad \hat{v}_2 = e_2 = (0,1,0,0)$

3) BASE DI IM: $w_1 = f(v_1) = \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix} \qquad w_2 = f(v_2) = \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix} \qquad w_3 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ BASI $\mathcal{E}$ DELLA FORMA CANONICA $\begin{cases} \text{IN PARTENZA: } v_1, v_2, v_3, v_4 \in \mathbb{R}^4 \\ \text{IN ARRIVO: } w_1, w_2, w_3 \in \mathbb{R}^3 \end{cases}$

$$\Rightarrow C = M_B^{-1} B N_B$$

PERTANTO:

$$C = M_B^{-1} B N_B = M_A^{-1} A N_A \Rightarrow B = M_B M_A^{-1} A N_A N_B^{-1}$$

$\Rightarrow \begin{cases} N_A N_B^{-1} \rightsquigarrow \text{MATRICE } 5\times 5 \text{ VETTORI BASE IN PARTENZA} \\ M_A M_B^{-1} \rightsquigarrow \text{MATRICE } 3\times 3 \text{ VETTORI BASE IN ARRIVO} \end{cases}$

$$M_B M_A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 5 & 1 & 0 \\ 10 & 0 & 1 \end{pmatrix} \quad M_A M_B^{-1} = \begin{pmatrix} -1/5 & 1/5 & 0 \\ 5/5 & -1/5 & 0 \\ 10/5 & -10/5 & 1 \end{pmatrix} \quad N_A N_B^{-1} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(columns of $M_A M_B^{-1}$: w_1, w_2, w_3; columns of $N_A N_B^{-1}$: v_1, v_2, v_3, v_1)

2) In ciascuna delle righe seguenti vengono presentate k matrici. Di queste, $(k-1)$ sono simili tra loro. Si richiede di determinare l'intrusa. Inoltre, per ogni coppia di matrici simili, non sarebbe male determinare la matrice di cambio di base (o meglio, una delle possibili matrici di cambio di base) che realizza la similitudine.

$A_{m\times m}$ È SIMILE A $B_{m\times m} \iff \exists\, M_{m\times m}$ INVERTIBILE T.C. $A = M^{-1}BM$

$\Rightarrow$ HANNO STESSO RANGO, DET, TRACCIA, AUTOVALORI

(a)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ \sqrt{3} & 2 \end{pmatrix} \quad \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} \quad E = \begin{pmatrix} 10 & 18 \\ -4 & -7 \end{pmatrix}$$

TRACCIA $\neq 3$ (fourth matrix crossed out)

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = M_B^{-1} \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} M_B \quad \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases} \quad (B - \lambda I)x = 0 \rightsquigarrow \text{AUTOVETTORI}$$

$$\lambda_1 = 1: \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} x = 0 \rightsquigarrow x_1 = (1,0) \qquad \lambda_2 = 2: \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} x = 0 \rightsquigarrow x_2 = (3,1)$$

$$\rightsquigarrow M_B = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

(M_B^{-1}, B, M_B)

$$C: (C - \lambda I)x = 0 \quad \lambda_1 = 1: \begin{pmatrix} 0 & 0 \\ \sqrt{3} & 1 \end{pmatrix} x = 0 \rightsquigarrow x_1 = (\sqrt{3}, -3)$$

$$\lambda_2 = 2: \begin{pmatrix} -1 & 0 \\ \sqrt{3} & 0 \end{pmatrix} x = 0 \rightsquigarrow x_2 = (0,1)$$

$$\rightsquigarrow M_C = \begin{pmatrix} \sqrt{3} & 0 \\ -3 & 1 \end{pmatrix}$$

$D: (D-\lambda I)x=0 \quad \lambda_1=1: \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} x=0 \rightsquigarrow x_1=(1,-1)$

$\lambda_2=2: \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} x=0 \rightsquigarrow x_2=(2,-1) \quad \rightsquigarrow M_D=\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$

$E: (E-\lambda I)x=0 \quad \lambda_1=1: \begin{pmatrix} 9 & 18 \\ -5 & -8 \end{pmatrix} x=0 \rightsquigarrow x_1=(2,-1)$

$\lambda_2=2: \begin{pmatrix} 8 & 18 \\ -5 & -9 \end{pmatrix} x=0 \rightsquigarrow x_2=(9;-5) \quad \rightsquigarrow M_E=\begin{pmatrix} 2 & 9 \\ -1 & -5 \end{pmatrix}$

$B \rightsquigarrow C \quad A=M_B^{-1}BM_B=M_C^{-1}CM_C \rightsquigarrow C=M_CM_B^{-1}BM_BM_C^{-1} \qquad M_{BC}=M_BM_C^{-1}$

IN GENERALE $\Rightarrow$ $\boxed{i \rightsquigarrow j \qquad M_{ij}=M_iM_j^{-1}}$

(l) ~~$\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}$~~ (DET ≠ 5) $\quad B\ \begin{pmatrix} 5 & 5 \\ -2 & -1 \end{pmatrix} \quad C\ \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \quad D\ \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \quad E\ \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad F\ \begin{pmatrix} 4 & -5 \\ 1 & 0 \end{pmatrix} \quad A\ \begin{pmatrix} ? & 0 \\ 0 & ? \end{pmatrix}$

$B=\begin{pmatrix} 5 & 5 \\ -2 & -1 \end{pmatrix}$ AUTOVALORI $\det(B-\lambda I)=\begin{vmatrix} 5-\lambda & 5 \\ -2 & -1-\lambda \end{vmatrix}=(5-\lambda)(-1-\lambda)+10=$

$=-5+\lambda^2+\lambda-5\lambda+10=\lambda^2-4\lambda+5=0 \rightsquigarrow \lambda=\dfrac{4\pm\sqrt{16-20}}{2}=2\pm i$

$\lambda_1=2+i: (B-\lambda I)x=0 \quad \begin{pmatrix} 3-i & 5 \\ -2 & -3-i \end{pmatrix} x=0 \quad x_1=(3+i,-2)$

$\lambda_2=2-i: (B-\lambda I)x=0 \quad \begin{pmatrix} 3+i & 5 \\ -2 & -3+i \end{pmatrix} x=0 \quad x_2=(3-i;-2) \quad \rightsquigarrow M_B=\begin{pmatrix} 3+i & 3-i \\ -2 & -2 \end{pmatrix}$

$\det(M_B)=-6-2i+6-2i=-4i \qquad M_B^{-1}=+\dfrac{1\,i}{4\cancel{i}}\begin{pmatrix} -2 & -3+i \\ 2 & 3+i \end{pmatrix}=\dfrac{1}{4}\begin{pmatrix} -2i & -1-3i \\ 2i & -1+3i \end{pmatrix}$

VER. $M_BM_B^{-1}=\dfrac{1}{4}\begin{pmatrix} \cancel{-6i}+2+\cancel{6i}+2 & \cancel{-3}\cancel{-3i}\cancel{-i}+\cancel{3}-3+\cancel{3i}+\cancel{i}+3 \\ +\cancel{3i}-\cancel{3i} & +2+\cancel{6i}+2-\cancel{6i} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A=M_B^{-1}BM_B=M_B^{-1}\begin{pmatrix} 15+5i-10 & 15-5i-10 \\ -6-2i+2 & -6+2i+2 \end{pmatrix}=M_B^{-1}\begin{pmatrix} 5+5i & 5-5i \\ -4-2i & -4+2i \end{pmatrix}=$

$=\dfrac{1}{4}\begin{pmatrix} -10i+10+4+2i+12i-6 & \cancel{-10i}\cancel{-10}+\cancel{4}-\cancel{2i}+\cancel{12i}+\cancel{6} \\ \cancel{10i}\cancel{-10}+\cancel{4}+\cancel{2i}-\cancel{12i}+\cancel{6} & 10i+10+4-2i-12i-6 \end{pmatrix}=\dfrac{1}{4}\begin{pmatrix} 8+4i & 0 \\ 0 & 8-4i \end{pmatrix}=\begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}$

OSS. n λ DISTINTI $\Leftrightarrow \exists$ BASE DI AUTOVETTORI $\Leftrightarrow \exists$ A DIAGONALE

$C=\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}$ $\quad \lambda_1=2+i \rightsquigarrow \begin{pmatrix}-i & -1\\ 1 & -i\end{pmatrix}X=0 \rightsquigarrow x_1=(i,1)$

$\lambda_2=2-i \rightsquigarrow \begin{pmatrix}i & -1\\ 1 & i\end{pmatrix}X=0 \rightsquigarrow x_2=(-i,1)$

$\rightsquigarrow M_C=\begin{pmatrix}i & -i\\ 1 & 1\end{pmatrix}$

$D=\begin{pmatrix}1 & -2\\ 1 & 3\end{pmatrix}$ $\quad \lambda_1=2+i \rightsquigarrow \begin{pmatrix}-1-i & -2\\ 1 & 1-i\end{pmatrix}X=0 \rightsquigarrow x_1=(1-i,-1)$

$\lambda_2=2-i \rightsquigarrow \begin{pmatrix}1+i & -2\\ 1 & 1+i\end{pmatrix}X=0 \rightsquigarrow x_2=(1+i,-1)$

$\rightsquigarrow M_D=\begin{pmatrix}1-i & 1+i\\ -1 & -1\end{pmatrix}$

$E=\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}$ $\quad \lambda_1=2+i \rightsquigarrow \begin{pmatrix}-i & 1\\ -1 & -i\end{pmatrix}X=0 \rightsquigarrow x_1=(-i,1)$

$\lambda_2=2-i \rightsquigarrow \begin{pmatrix}i & 1\\ -1 & i\end{pmatrix}X=0 \rightsquigarrow x_2=(i,1)$

$\rightsquigarrow M_E=\begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}$

$F=\begin{pmatrix}4 & -5\\ -1 & 0\end{pmatrix}$ $\quad \lambda_1=2+i \rightsquigarrow \begin{pmatrix}2-i & -5\\ -1 & -2-i\end{pmatrix}X=0 \rightsquigarrow x_1=(2+i,-1)$

$\lambda_2=2-i \rightsquigarrow \begin{pmatrix}2+i & -5\\ -1 & -2+i\end{pmatrix}X=0 \rightsquigarrow x_2=(-2+i,1)$

$\rightsquigarrow M_F=\begin{pmatrix}2+i & -2+i\\ -1 & 1\end{pmatrix}$

(c)

$A=J_{2\times 2}$	B	C	D	$MG=2$	E
$\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$	$\begin{pmatrix}2 & 2\\ 0 & 2\end{pmatrix}$	$\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}$	$\begin{pmatrix}6 & -4\\ 4 & -2\end{pmatrix}$	~~$\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$~~	$\begin{pmatrix}2 & 0\\ 7 & 2\end{pmatrix}$ $\lambda_1=\lambda_2=2$, $MG=1$

$B=\begin{pmatrix}2 & 2\\ 0 & 2\end{pmatrix}$ $\quad \lambda=2 \rightsquigarrow \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}X=0 \rightsquigarrow x_1=(1,0)$ $\quad MG=1 < MA=2$ JORDANIZZ.

$Ax=2x+x_1 \rightsquigarrow \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}X=\begin{pmatrix}1\\ 0\end{pmatrix} \rightsquigarrow x_2=(0,1/2)$ $\quad \rightsquigarrow M_B=\begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix}$

$M_B^{-1}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ $\quad A=J_{2\times 2}=M_B^{-1}BM_B=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$

$C=\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}$ $\quad \lambda=2 \rightsquigarrow \begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}X=0 \rightsquigarrow x_1=(1,-1)$

$Ax=2x+x_1 \rightsquigarrow \begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}X=\begin{pmatrix}1\\ -1\end{pmatrix} \rightsquigarrow x_2=(-1,0)$ $\quad \rightsquigarrow M_C=\begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}$

$M_C^{-1}=\begin{pmatrix}0 & -1\\ -1 & -1\end{pmatrix}$ $\quad A=J_{2\times 2}=M_C^{-1}CM_C=\begin{pmatrix}-1 & -3\\ -2 & -2\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$

$$D=\begin{pmatrix}6&-5\\5&-2\end{pmatrix}\quad \lambda=2 \rightsquigarrow \begin{pmatrix}5&-5\\5&-5\end{pmatrix}X=0 \rightsquigarrow X_1=(1,1)$$

$$Ax=2x+X_1 \rightsquigarrow \begin{pmatrix}5&-5\\5&-5\end{pmatrix}X=\begin{pmatrix}1\\1\end{pmatrix} \rightsquigarrow X_2=(5/5,1)$$

$$\rightsquigarrow M_D=\begin{pmatrix}1&5/5\\1&1\end{pmatrix}$$

$$E=\begin{pmatrix}2&0\\7&2\end{pmatrix}\quad \lambda=2 \rightsquigarrow \begin{pmatrix}0&0\\7&0\end{pmatrix}X=0 \rightsquigarrow X_1=(0,1)$$

$$Ax=2x+X_1 \rightsquigarrow \begin{pmatrix}0&0\\7&0\end{pmatrix}X=\begin{pmatrix}0\\1\end{pmatrix} \rightsquigarrow X_2=(1/7,0)$$

$$\rightsquigarrow M_E=\begin{pmatrix}0&1/7\\1&0\end{pmatrix}$$

(d)

$$A=\begin{pmatrix}1&0&0\\0&3&0\\0&0&-2\end{pmatrix}\quad B=\begin{pmatrix}3&0&0\\0&-2&0\\0&0&1\end{pmatrix}\quad C=\begin{pmatrix}1&0&0\\6&-2&0\\5&4&3\end{pmatrix}\quad \begin{pmatrix}-3&0&0\\0&2&0\\0&0&-1\end{pmatrix}$$

TRACCIA ≠ 2

$\lambda_1=1$
$\lambda_2=3$
$\lambda_3=-2$

$$B=\begin{pmatrix}3&0&0\\0&-2&0\\0&0&1\end{pmatrix}\quad \lambda_1=1 \rightsquigarrow \begin{pmatrix}2&0&0\\0&-3&0\\0&0&0\end{pmatrix}X=0 \rightsquigarrow X_1=\begin{pmatrix}0\\0\\1\end{pmatrix}\quad \lambda_2=3 \rightsquigarrow \begin{pmatrix}0&0&0\\0&-5&0\\0&0&-2\end{pmatrix}X=0 \rightsquigarrow X_2=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\lambda_3=-2 \rightsquigarrow \begin{pmatrix}5&0&0\\0&0&0\\0&0&3\end{pmatrix}X=0 \rightsquigarrow X_3=\begin{pmatrix}0\\1\\0\end{pmatrix} \rightsquigarrow M_B=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad M_B^{-1}=M_B^{T}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

(columns of M_B: x_1, x_2, x_3)

SCAMBIA COLONNE: $c_1\to c_2$, $c_2\to c_3$, $c_3\to c_1$

SCAMBIA RIGHE: $R_1\to R_2$, $R_2\to R_3$, $R_3\to R_1$

$$\rightsquigarrow M_B^{-1}BM_B=M_B^{-1}\begin{pmatrix}0&3&0\\0&0&-2\\1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&3&0\\0&0&-2\end{pmatrix}=A$$

(columns of the middle matrix: c_3, c_1, c_2; rows of the result: R_3, R_1, R_2)

$$C=\begin{pmatrix}1&0&0\\6&-2&0\\5&5&3\end{pmatrix}\quad \lambda_1=1 \rightsquigarrow \begin{pmatrix}0&0&0\\6&-3&0\\5&5&2\end{pmatrix}X=0 \rightsquigarrow X_1=\begin{pmatrix}1\\2\\-\frac{13}{2}\end{pmatrix}\quad \lambda_2=3 \rightsquigarrow \begin{pmatrix}-2&0&0\\6&-5&0\\5&5&0\end{pmatrix}X=0 \rightsquigarrow X_2=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\lambda_3=-2 \rightsquigarrow \begin{pmatrix}3&0&0\\6&0&0\\5&5&5\end{pmatrix}X=0 \rightsquigarrow X_3=\begin{pmatrix}0\\1\\-5/5\end{pmatrix} \rightsquigarrow M_C=\begin{pmatrix}1&0&0\\2&0&1\\-\frac{13}{2}&1&-\frac{5}{5}\end{pmatrix}$$

(e) $\begin{pmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ -3 & 0 & 1 \end{pmatrix}$ TRACCIA ≠ 5 $\quad$ B $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad$ C $\begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad$ D = J $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ JORDANIZZ. $\quad$ E $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 7 & 5 & 1 \end{pmatrix}$ $\lambda_1=\lambda_2=2$, $\lambda_3=1$

$$B:\ \lambda_1=\lambda_2=2 \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} X=0 \rightsquigarrow X_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad X_2=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

MA = MG = 2 ⇒ DIAGONALIZZ.

$$X_3=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow M_B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad M_B^{-1}=M_B^T=M_B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rightsquigarrow A=M_B^{-1}BM_B=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C:\ \lambda_1=\lambda_2=2 \rightsquigarrow \begin{pmatrix} 0 & -1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} X=0 \rightsquigarrow X_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad X_2=\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \lambda_3=1 \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} X=0 \rightsquigarrow X_3=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow M_C=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad M_C^{-1}=\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \rightsquigarrow M_C^{-1}CM_C=A$$

$$D:\ \lambda_1=\lambda_2=2 \rightsquigarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} X=0 \rightsquigarrow X_1=\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad X_2=\ ? \rightsquigarrow$$

MG = 1 < MA = 2
NON SIMILE AD A
NON DIAGONALIZZABILE
MA JORDANIZZABILE

$$Ax=2x+X_1 \rightsquigarrow \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} X=\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \rightsquigarrow X_2=\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \quad \lambda_3=1 \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & 0 \end{pmatrix} X=0 \quad X_3=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow M_D=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix} \quad M_D^{-1}=\begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rightsquigarrow M_D^{-1}DM_D=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E:\ \lambda_1=\lambda_2=2 \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 7 & 5 & -1 \end{pmatrix} X=0 \rightsquigarrow X_1=\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} \quad X_2=\begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} \quad \lambda_3=1 \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 5 & 0 \end{pmatrix} X=0 \rightsquigarrow X_3=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow M_E=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & 7 & 1 \end{pmatrix} \quad M_E^{-1}=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -7 & -5 & 1 \end{pmatrix} \rightsquigarrow M_E^{-1}EM_E=A$$

(1)

$$B=\begin{pmatrix}1&0&2\\0&2&0\\0&0&1\end{pmatrix}\quad C=\begin{pmatrix}1&0&1\\0&1&0\\0&0&2\end{pmatrix}\quad D=\begin{pmatrix}2&3&4\\0&2&1\\0&1&0\end{pmatrix}\quad A\equiv J=\begin{pmatrix}1&1&0\\0&1&0\\0&0&2\end{pmatrix}\quad E=\begin{pmatrix}1&1&0\\0&1&1\\0&0&2\end{pmatrix}$$

C: $MG(1)=MA(1)=2$ D: $DET\neq 2$ $\lambda_1=\lambda_2=1$, $\lambda_3=2$

B: $\lambda_1=\lambda_2=1 \rightsquigarrow \begin{pmatrix}0&0&2\\0&1&0\\0&0&0\end{pmatrix}X=0 \quad x_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $MG=1<MA=2$ JORDANIZZ. $\begin{pmatrix}0&0&2\\0&1&0\\0&0&0\end{pmatrix}X=\begin{pmatrix}1\\0\\0\end{pmatrix} \rightsquigarrow x_2=\begin{pmatrix}0\\0\\1/2\end{pmatrix}$

$\lambda_3=2 \rightsquigarrow \begin{pmatrix}-1&0&2\\0&0&0\\0&0&-1\end{pmatrix}X=0 \quad x_3=\begin{pmatrix}0\\1\\0\end{pmatrix}$ $M_B=\begin{pmatrix}1&0&0\\0&0&1\\0&1/2&0\end{pmatrix} \rightsquigarrow M_B^{-1}BM_B=\begin{pmatrix}1&1&0\\0&1&0\\0&0&2\end{pmatrix}=A$

C: $\lambda_1=\lambda_2=1 \rightsquigarrow \begin{pmatrix}0&0&1\\0&0&0\\0&0&1\end{pmatrix}X=0 \quad x_1=\begin{pmatrix}1\\0\\0\end{pmatrix} \quad x_2=\begin{pmatrix}0\\1\\0\end{pmatrix} \rightsquigarrow$ $MG=2=MA$ DIAGONALIZZ. INTRUSA 1 $\rightsquigarrow \begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}$

D: $\lambda_1=\lambda_2=1 \rightsquigarrow \begin{pmatrix}1&3&5\\0&1&1\\0&1&-1\end{pmatrix}X=0$ RANGO 3 → ??? $DET(D)=2\begin{vmatrix}2&1\\1&0\end{vmatrix}=-2\neq 2$ INTRUSA 2

E: $\lambda_1=\lambda_2=1 \rightsquigarrow \begin{pmatrix}0&1&0\\0&0&1\\0&0&1\end{pmatrix}X=0 \quad x_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $MG=1<MA=2$ JORDANIZZ. $\begin{pmatrix}0&1&0\\0&0&1\\0&0&1\end{pmatrix}X=\begin{pmatrix}1\\0\\0\end{pmatrix} \rightsquigarrow x_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\lambda_3=2 \rightsquigarrow \begin{pmatrix}-1&1&0\\0&-1&1\\0&0&0\end{pmatrix}X=0 \quad x_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $M_E=\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix} \rightsquigarrow M_E^{-1}EM_E=A$

B → E: $A=M_B^{-1}BM_B=M_E^{-1}EM_E \rightsquigarrow E=M_EM_B^{-1}BM_BM_E^{-1}=M_{BE}^{-1}BM_{BE}$

$$M_{BE}=M_BM_E^{-1}=\begin{pmatrix}1&0&0\\0&0&1\\0&1/2&0\end{pmatrix}\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&-1\\0&0&1\\0&1/2&-1/2\end{pmatrix} \qquad M_{BE}^{-1}=\begin{pmatrix}1&1&0\\0&1&2\\0&1&0\end{pmatrix}$$

(g) $\underset{A\equiv J}{\begin{pmatrix}3&1&0\\0&3&1\\0&0&3\end{pmatrix}}$ $\underset{B}{\begin{pmatrix}3&1&1\\0&3&1\\0&0&3\end{pmatrix}}$ $\underset{C}{\begin{pmatrix}3&3&3\\0&3&3\\0&0&3\end{pmatrix}}$ $\underset{D\equiv\hat{J}}{\begin{pmatrix}3&0&1\\0&3&1\\0&0&3\end{pmatrix}}$ $\underset{E}{\begin{pmatrix}3&0&0\\-1&3&0\\0&1&3\end{pmatrix}}$ $\lambda_1=\lambda_2=\lambda_3=3$

B: $\lambda=3 \rightsquigarrow \begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}X=0 \rightsquigarrow X_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ MG=1 < MA=3, JORDANIZZ. 1 BLOCCO, $AX=3X+X_1 \rightsquigarrow$ $\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}X=\overset{X_1}{\begin{pmatrix}1\\0\\0\end{pmatrix}} \rightsquigarrow X_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$

$AX=3X+X_2$ $\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}X=\overset{X_2}{\begin{pmatrix}0\\1\\0\end{pmatrix}} \rightsquigarrow X_3=\begin{pmatrix}0\\-1\\1\end{pmatrix} \rightsquigarrow M_B=\begin{pmatrix}1&0&0\\0&1&-1\\0&0&1\end{pmatrix} \rightsquigarrow M_B^{-1}BM_B=A$

C: $\lambda=3 \rightsquigarrow \begin{pmatrix}0&3&3\\0&0&3\\0&0&0\end{pmatrix}X=0 \rightsquigarrow X_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ MG=1 < MA=3, JORDANIZZ. 1 BLOCCO, $AX=3X+X_1 \rightsquigarrow$ $\begin{pmatrix}0&3&3\\0&0&3\\0&0&0\end{pmatrix}X=\overset{X_1}{\begin{pmatrix}1\\0\\0\end{pmatrix}} \rightsquigarrow X_2=\begin{pmatrix}0\\1/3\\0\end{pmatrix}$

$AX=3X+X_2$ $\begin{pmatrix}0&3&3\\0&0&3\\0&0&0\end{pmatrix}X=\overset{X_2}{\begin{pmatrix}0\\1/3\\0\end{pmatrix}} \rightsquigarrow X_3=\begin{pmatrix}0\\-1/9\\1/9\end{pmatrix} \rightsquigarrow M_C=\begin{pmatrix}1&0&0\\0&1/3&-1/9\\0&0&1/9\end{pmatrix} \rightsquigarrow M_C^{-1}CM_C=A$

D: $\lambda=3 \rightsquigarrow \begin{pmatrix}0&0&1\\0&0&1\\0&0&0\end{pmatrix}X=0 \rightsquigarrow$ MG=2 < MA=3, JORDANIZZ. 2 BLOCCHI $\left(\begin{array}{cc|c}3&1&0\\0&3&0\\\hline 0&0&3\end{array}\right) \rightsquigarrow \begin{pmatrix}0&0&1\\0&0&1\\0&0&0\end{pmatrix}X=0 \rightsquigarrow X_1=\begin{pmatrix}1\\1\\0\end{pmatrix}$ SCELTA OBBLIG. PER TROV. X_2

$AX=3X+X_1 \rightsquigarrow \begin{pmatrix}0&0&1\\0&0&1\\0&0&0\end{pmatrix}X=\overset{X_1}{\begin{pmatrix}1\\1\\0\end{pmatrix}} \rightsquigarrow X_2=\begin{pmatrix}0\\0\\1\end{pmatrix}$ $X_3=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\rightarrow$ MA FUNZIONEREBBE ANCHE $X_3=e_2$

$\rightsquigarrow M_D=\begin{pmatrix}1&0&1\\1&0&0\\0&1&0\end{pmatrix}$ $M_D^{-1}=\begin{pmatrix}0&1&0\\0&0&1\\1&-1&0\end{pmatrix} \rightsquigarrow M_D^{-1}DM_D=\begin{pmatrix}3&1&0\\0&3&0\\0&0&3\end{pmatrix}=\hat{A}\equiv\hat{J}$

E: $\lambda=3 \rightsquigarrow \begin{pmatrix}0&0&0\\-1&0&0\\0&1&0\end{pmatrix}X=0$ MG=1 < MA=3, JORDANIZZ. 1 BLOCCO, $AX=3X+X_1 \rightsquigarrow$ $\rightsquigarrow X_1=\begin{pmatrix}0\\0\\1\end{pmatrix} \rightsquigarrow X_2=\begin{pmatrix}0\\1\\0\end{pmatrix} \rightsquigarrow X_3=\begin{pmatrix}-1\\0\\0\end{pmatrix}$

$\rightsquigarrow M_E=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}$ $M_E^{-1}=\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix} \rightsquigarrow M_E^{-1}EM_E=A$