1. Determine whether the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 + \dot{u}u + u^2 + u\right)\, dx$$

has the minimum in the class $C^1([0,1])$.

See CdV 2019_6 Ex 1

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2. For every positive integer d, let us consider the following three inequalities:

$$\int_{\mathbb{R}^d} u(x)^{32}\, dx \le K_d, \qquad u(0) \le K_d, \qquad \|\nabla u(0)\| \le K_d,$$

(1) (2) (3)

For each of them, determine the values of d for which there exists a constant K_d that makes it true for every $u \in C_c^\infty(\mathbb{R}^d)$ whose norm in $W^{20,19}(\mathbb{R}^d)$ is less than or equal to 1.

(1) The inequality is equivalent to the embedding

$$W^{20,19}(\mathbb{R}^d) \longrightarrow L^{32}(\mathbb{R}^d)$$

Now we know that $W^{m,p}(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d)$ for every $p \le q \le p^* = \frac{pd}{d-mp}$, which in this case means $19 \le 32 \le \frac{19d}{d-20\cdot 19}$

The last inequality holds true $\iff d \le \frac{32\cdot 380}{13} \iff \boxed{d \le 935}$

(The cases where $d \le 380$ are easy).

(2) The inequality is equivalent to the embedding

$$W^{20,19}(\mathbb{R}^d) \longrightarrow L^\infty(\mathbb{R}^d).$$

This holds true $\iff d < 20\cdot 19 \iff \boxed{d \le 379}$

(3) The inequality is equivalent to the embedding

$$W^{20,19}(\mathbb{R}^d) \longrightarrow W^{1,\infty}(\mathbb{R}^d).$$

This holds true $\iff d < 19\cdot 19 \iff \boxed{d \le 360}$

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3. Let us consider the square $\Omega = (0,1)^2$, and for every real number $\varepsilon > 0$ let us set

$$I(\varepsilon) := \inf\left\{ \underbrace{\int_\Omega (u_x^2 + u_y^2 + u^7)\,dx\,dy}_{F(u)} : u \in C_c^1(\Omega),\ \int_\Omega (u_x^2 + 5u_y^2)\,dx\,dy \le \varepsilon \right\}.$$

(a) Prove that $I(\varepsilon)$ is a real number for every $\varepsilon > 0$.

(b) Determine whether there exists $\varepsilon > 0$ such that $I(\varepsilon) = 0$.

(c) Find the limit of $I(\varepsilon)$ as $\varepsilon \to +\infty$.

(a) Standard direct method. Take a minimizing sequence $\{u_n\}$. The integral constraint and DBC provide compactness, namely

$u_{n_k} \longrightarrow u_\infty$ L^2 and $\nabla u_{n_k} \rightharpoonup \nabla u_\infty$ weak L^2.

Due to Sobolev embeddings we know that $u_{n_k} \to u_\infty$ in L^7. From the LSC of the norm we conclude that u_∞ satisfies the integral constraint, and $\liminf F(u_{n_k}) \ge F(u_\infty)$. Since the min. exists in $H_0^1(\Omega)$, the infimum is finite.

(b) $\boxed{\text{YES}}$ $I(\varepsilon) = 0$ when ε is small enough. Indeed

$$\|u\|_{L^7} \underset{\text{Sobolev embedding}}{\le} c_1 \left(\|u\|_{L^2} + \|\nabla u\|_{L^2} \right) \underset{\text{Poincaré}}{\le} c_2 \|\nabla u\|_{L^2}$$

and therefore

$$F(u) \ge \|\nabla u\|_{L^2}^2 - c_2^7 \|\nabla u\|_{L^2}^7.$$

When ε is small also $\|\nabla u\|_{L^2}$ is small, and therefore the RHS is ≥ 0, which proves that the LHS is ≥ 0.
On the other hand $u \equiv 0$ is always a competitor.

(c) $\boxed{I(\varepsilon) \to -\infty}$ as $\varepsilon \to +\infty$. Let $u \in C_c^1(\Omega)$ be a function ≤ 0 and $\not\equiv 0$. Set $u_n(x,y) = n\,u(x,y)$.
Then it turns out that

$$F(u_n) \longrightarrow -\infty \quad \text{as } n \to +\infty.$$

On the other hand, for every $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that u_n satisfies the integral constraint with ε, and hence u_n is a competitor in the definition of $I(\varepsilon)$.

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4. For every $f : (0,1) \to \mathbb{R}$, let us set

$$[Tf](x) := f(x^2) \qquad \forall x \in (0,1).$$

Determine whether the restriction of T defines

(a) a *continuous* operator $H^1((0,1)) \to L^4((0,1))$,
(b) a *continuous* operator $W^{1,4}((0,1)) \to W^{1,4}((0,1))$,
(c) a *compact* operator $H^1((0,1)) \to H^1((0,1))$.

(a) $\boxed{\text{YES}}$ Indeed

$f_n \to f_\infty$ in H^1 $\Rightarrow$ $f_n \to f_\infty$ unif
$\Rightarrow$ $f_n(x^2) \to f_\infty(x^2)$ unif. $\Rightarrow$ $f_n(x^2) \to f_\infty(x^2)$ in L^4

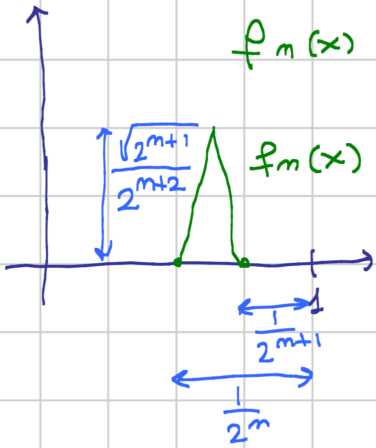
(b) $\boxed{\text{YES}}$ Indeed

$$\| f(x^2) \|_{L^4} \le \| f(x^2) \|_{C^0} = \| f(x) \|_{C^0} \le C_1 \| f(x) \|_{W^{1,4}}$$

$$\int_0^1 \left| \frac{d}{dx} f(x^2) \right|^4 = \int_0^1 16 x^4 f'(x^2)^4 \, dx \overset{y = x^2}{=} 8 \int_0^1 f'(y)^4 \, y^{3/2} \le 8 \int_0^1 f'(y)^4 dy$$

This is enough to prove that the map is well posed and continuous.

(c) $\boxed{\text{NO}}$ Let $\{f_n\} \subseteq H^1$ be as in the figure.



One can check that

- $\{f_n\}$ is bounded in H^1,
- the supports are disjoint,
- $\| f_n' \|_{L^2} = 1$ for every n.

On the other hand

$$\int_0^1 |2x f_n'(x^2) - 2x f_m'(x^2)|^2 dx \overset{y = x^2}{=} 2 \int_0^1 \sqrt{y} \, |f_n'(y) - f_m'(y)|^2 dy$$

$$\underset{\substack{y \ge \frac{1}{4} \\ \text{for } n, m \text{ large}}}{\ge} \int_0^1 |f_n'(y) - f_m'(y)|^2 \underset{\substack{n \neq m \\ \text{disjoint supports}}}{\ge} \int_0^1 f_n'(y)^2 + \int_0^1 f_m'(y)^2 = 2$$

which proves that $\{f_n(x^2)\}$ has no Cauchy subsequence in H^1.

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