1. Let us consider the functionals

$$F(u) = u(0) + \int_0^1 (\dot{u}^2 + u^2)\,dx, \qquad G(u) = [u(0)]^3 + \int_0^1 (\dot{u}^2 + u^2)\,dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary condition $u(1) = 3$.

(b) Discuss the minimum problem for $G(u)$ with boundary condition $u(1) = 3$.

(a) $F(u+tv) = u(0) + tv(0) + \int_0^1 (\dot{u}^2+u^2) + 2t\int_0^1 (\dot{u}\dot{v} + uv) + t^2 \int_0^1 (\dot{v}^2+v^2)$

Therefore, any min. point $u_0(x)$ satisfies

$$\delta F(u_0, v) = v(0) + 2\int_0^1 (\dot{u}_0\dot{v} + u_0 v_0) = 0$$

Integrating by parts we obtain that

$$v(0) + 2\dot{u}_0(1)\,v(1) - 2\dot{u}_0(0)\,v(0) + 2\int_0^1 (-\ddot{u}_0 + u_0)\,v = 0$$

From this relation we deduce in the usual way that $u_0$ satisfies

$$\begin{cases} \ddot{u}_0 = u_0 \\ u_0(1) = 3 \\ \dot{u}_0(0) = \frac{1}{2} \end{cases}$$

$u_0(x) = a\cosh x + b\sinh x$ $\qquad$ $\dot{u}_0(x) = a\sinh x + b\cosh x$

$a\cosh 1 + b\sinh 1 = 3$

$b = \frac{1}{2} \rightsquigarrow a = \dfrac{3 - \frac{1}{2}\sinh 1}{\cosh 1}$

The system has a unique solution $u_0(x)$. This is the unique min. point, as we can prove by computing $F(u_0 + v)$ where $v(x)$ is any admissible variation, namely $v \in C^1([0,1])$ with $v(1) = 0$.

↑ or even $H^1((0,1))$

(b) The minimum does NOT exist, and the infimum is $-\infty$.

A possible minimizing sequence is $u_n(x) = n(x-1) + 3$, because

$$G(u_n) = (3-n)^3 + O(n^2) \quad \text{as } n \to +\infty.$$

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Alternative for point (a): observe that $\int_0^1 \dot{u} = u(1) - u(0) = 3 - u(0)$, so that

$F(u) = 3 + \int_0^1 (\dot{u}^2 - \dot{u} + u^2)$. Then proceed with this new functional.

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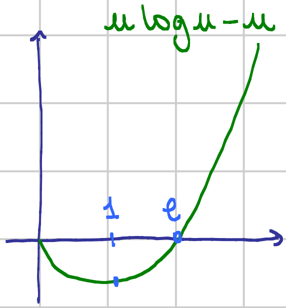
2. Let $a$ be a positive real number, and let us consider the boundary value problem

$$u'' = \log u, \qquad u(0) = u(2020) = a.$$

(a) Discuss existence, uniqueness and regularity of solutions.

(b) Determine the values of $a$ for which solutions are less than 1 for every $x \in [0, 2020]$.




(a) Let us consider the functional

$$F(u) := \int_0^{2020} \Big\{ \frac{1}{2}\dot{u}^2 + \underbrace{u \log u - u}_{\text{antiderivative of } \log u} \Big\} dx$$

and the problem

$$\min \{ F(u) : u \in H^1((0,2020)),\ u(0) = u(2020) = a,\ \underbrace{u(x) \geq 0 \quad \forall x \in [0,2020]}_{\text{ESSENTIAL!!}} \}$$

- Existence. This follows from the direct method in the usual way. In particular, compactness follows from estimates such as

$$\frac{1}{2}p^2 + s\log s - s \geq \frac{1}{2}p^2 - A \qquad \forall p \in \mathbb{R} \quad \forall s \geq 0,$$

while lower semicontinuity follows from the convexity of the Lagrangian wrt $p$ and the continuity wrt $s$.

- Regularity The claim is that any min. point is a sol. to the given equ. To this end, it is enough to compute ELE in the usual way. Achtung! Before computing ELE, we need to know a priori that $u(x) > 0$ for every $x \in [0,2020]$ (if not, $u(t) + t v(x)$ might be negative!)

- Truncation argument. With a standard truncation argument, one can show that

  → if $a \in (0,1]$, then $a \leq u(x) \leq 1$ for every $x \in [0,2020]$,

  → if $a \geq 1$, then $1 \leq u(x) \leq a$ " " " .

  In both cases, this is enough to justify the computation of ELE.

- Uniqueness This follows in the usual way from two observations.

  → Every solution to the diff. equ. is a minimum point.

  → The solution to the min. pbm. is unique.

Both facts follow from the strict convexity of the Lagrangian wrt $(s,p)$.

(a) It turns out that $u(x) < 1$ for every $x \in [0,2020] \iff \boxed{a \in (0,1)}$

Indeed, from the truncation argument we already know that $a \leq u(x) \leq 1$ in $[0,2020]$. If $u(x_0) = 1$ for some $x_0 \in (0,2020)$, then $x_0$ is a max. point, and hence $\dot{u}(x_0) = 0$. But then the given equ. with Cauchy data $u(x_0) = 1$, $\dot{u}(x_0) = 0$ has at least two solutions.

3. Let $B$ denote an open ball in $\mathbb{R}^3$. For every real number $p > 1$, let us set

$$S(p) := \sup \left\{ \int_B u^5\,dx : u \in C^\infty(B),\ \int_B |\nabla u|^p\,dx = \int_B u\,dx = 5 \right\}.$$

(a) Determine whether there exists $p_0 < 2$ such that $S(p)$ is a real number for every $p \geq p_0$.

(b) Determine whether there exists a real number $p > 1$ such that $S(p) = +\infty$.

(c) Determine whether there exists a real number $M$ such that $S(p) \leq M$ for every $p \geq 2$.

(a) $\boxed{\text{YES}}$ Indeed $S(p)$ is a real number for every $p \geq \frac{15}{8}$, namely the exponent such that $W^{1,p} \to L^5$. Indeed, if $m$ denotes the average of $u$ in $B$,

$$\int_B u^5 \leq \int_B |u|^5 = \|u\|_{L^5}^5 \leq (\|u-m\|_{L^5} + \|m\|_{L^5})^5$$

$$\underset{(a+b)^5 \leq \text{cost}\,(a^5+b^5)}{\leq} \text{cost}\,\|u-m\|_{L^5}^5 + \text{cost} \underset{\text{Poincaré-Sobolev-Wirtinger}}{\leq} \text{cost}\,\|\nabla u\|_{L^p}^5 + \text{cost} \underset{\text{integral constraint on } \nabla u}{\leq} \text{cost}$$

(b) $\boxed{\text{YES}}$ $S(p) = +\infty$ for every $p < \frac{12}{7}$ (the exponent of the embedding $W^{1,p} \to L^4$). Indeed, let $\varphi \in C_c^\infty(\mathbb{R}^3)$ be an $\boxed{\text{ODD}}$ (nontrivial) function, and let us observe that $u_n(x) := \frac{5}{\text{meas}(B)} + n^a\varphi(nx)$ satisfies $\int_B u_n = 5$ when $n$ is large enough (wlog $B$ has center in the origin).

If $p(a+1) = 3$, then $\int_B |\nabla u_n|^p$ does NOT depend on $p$ (so we can assume it is 5 up to multiplying $\varphi$ by a suitable constant). On the other hand,

$$\int_B u_n^5 = \int_B (5 + n^a\varphi(nx))^5 \underset{\text{Newton}}{=} \underbrace{\int_B n^{5a}\varphi(nx)^5}_{=0} + \underbrace{\int_B 25\,n^{4a}\varphi(nx)^4}_{=\text{cost}\cdot n^{4a-3}\cdot\int_{\mathbb{R}^3}\varphi(y)^4\,dy} + \underbrace{O(n^{3a})}_{=O(n^{3a-3})}$$

If $p < \frac{12}{7}$, then $a = \frac{3}{p} - 1$ satisfies $4a - 3 > 0$, and hence $\int_B u_n^5 \to +\infty$.

(c) $\boxed{\text{YES}}$ From the inequality $x^2 \leq x^p + 1$ (for every $x \geq 0$) we deduce that

$$\int_B |\nabla u|^2 \leq \int_B |\nabla u|^p + \text{meas}(B) \leq K$$

and therefore $$S(p) \leq \sup\left\{ \int_B u^5 : \int_B u = 5,\ \int_B |\nabla u|^2 \leq K \right\}$$

The RHS is finite because of the same argument of point (a) (because $\frac{15}{8} < 2$).

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Remark Actually $S(p) = +\infty$ for every $p < \frac{15}{8}$. One can choose $u(x) = a + \frac{b}{|x|^c}$ such that $\int u^5 = +\infty$ and $\int u = \int |\nabla u|^p = 5$, and conclude with approx. argument.

4. For every sequence $\{x_n\}$ of real numbers, let us set

$$C(x_1, x_2, x_3, \ldots) = \left( x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots \right).$$

In other words, $C(\{x_n\})$ is the sequence $\{y_n\}$ with

$$y_n := \frac{1}{n} \sum_{i=1}^{n} x_i \qquad \forall n \geq 1.$$

Determine whether the restriction of $C$ defines

(a) a bounded operator $\ell^1 \to \ell^1$,
(b) a bounded operator $\ell^1 \to \ell^2$,
(c) a bounded operator $c \to c$ (as usual $c$ denotes the space of sequences with a finite limit),
(d) a compact operator $\ell^\infty \to \ell^\infty$.

First of all, we observe that $C$ is linear, and therefore continuity is equivalent to Lip. continuity.

(a) $\boxed{\text{NO}}$ It is not even well-defined, because $C(1,0,0,\ldots) \notin \ell^1$

(b) $\boxed{\text{YES}}$ Indeed $|y_i| \leq \frac{1}{i} \|x\|_{\ell^1}$ for every $i \geq 1$, and hence

$$\|C(x)\|_{\ell^2} = \left\{ \sum_{i=1}^{\infty} y_i^2 \right\}^{1/2} \leq \|x\|_{\ell^1} \cdot \left\{ \sum_{i=1}^{\infty} \frac{1}{i^2} \right\}^{1/2}$$

convergent ☺

(c) $\boxed{\text{YES}}$ Indeed $|y_i| \leq \|x\|_{\ell^\infty}$, and therefore $C: L^\infty \to L^\infty$ is 1-Lip. continuous. Since $C(x) \in c$ for every $x \in c$, this implies also that $C: c \to c$ is 1-Lip. continuous.

(d) $\boxed{\text{NO}}$ Let $s_n \in \ell^\infty$ be the sequence $s_n = (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, \underbrace{0, 0, \ldots}_{\text{all } 0\text{'s}})$

It is clear that $\{s_n\} \subseteq \ell^\infty$ is bounded

It is also clear that every component of $C(s_n) \to 1$ as $n \to +\infty$ (and actually is equal to 1 for $n$ large). Therefore, a subsequence $C(s_{n_k})$, if convergent in $\ell^\infty$, converges to the sequence $(1, 1, 1, \ldots)$

all 1's

On the other hand, $C(s_n)$ converges to 0 for every fixed $n$, and the limit is stable with respect to $\ell^\infty$ convergence.

(Please pay attention: we are dealing with "sequences of sequences").