1. Determine whether the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 + \dot{u}u + u^2 + u\right)\,dx$$

has the minimum in the class $C^1([0,1])$.

ELE: $(2\dot{u} + u)' = \dot{u} + 2u + 1 \rightsquigarrow 2\ddot{u} + \cancel{\dot{u}} = \cancel{\dot{u}} + 2u + 1$

$\ddot{u} = u + \frac{1}{2} \rightsquigarrow u(x) = -\frac{1}{2} + a\cosh x + b\sinh x$

$\dot{u}(x) = a\sinh x + b\cosh x$

BCs: $2\dot{u} + u = 0$ for $x \in \{0,1\}$

$x = 0$: $2b - \frac{1}{2} + a = 0$

$x = 1$: $2a\sinh 1 + 2b\cosh 1 - \frac{1}{2} + a\cosh 1 + b\sinh 1 = 0$

$$\begin{cases} a + 2b = \frac{1}{2} \\ (2\sinh 1 + \cosh 1)\,a + (2\cosh 1 + \sinh 1)\,b = \frac{1}{2} \end{cases}$$

$\text{Det} = -3\sinh 1 \neq 0$

The solution is unique

The corresponding $u_0(x)$ is the $\boxed{\text{unique minimum point}}$

Indeed

$$F(u_0 + v) = F(u_0) + \underbrace{\int_0^1 \left[(2\dot{u}_0 + u_0)\dot{v} + (\dot{u}_0 + 2u_0 + 1)v\right]}_{=0 \text{ because of ELE + BCs}} + \underbrace{\int_0^1 (\dot{v}^2 + v^2 + v\dot{v})}_{\substack{\geq 0 \\ \text{(quadratic form)}}}$$

$$\geq F(u_0)$$

with equality if and only if $\dot{v}^2 + v^2 + v\dot{v} = 0$ in $[0,1]$. Since the quadratic form is positive, this happens if and only if $v \equiv 0$.

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2. Let us consider the boundary value problem

$$u'' = -\frac{x^3}{u^3}, \qquad u(0) = 1, \quad u(2) = 3.$$

(a) Discuss existence, uniqueness and regularity of positive solutions.

(b) Determine the minimum of the solution in the interval $[0, 2]$.

(a) Let us consider the minimum problem

$$\min\left\{ \underbrace{\int_0^2 \left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}\frac{x^3}{u^2}\right)dx}_{F(u)} : u \in H^1((0,2)),\ u \geq 0,\ \text{DBC} \right\}$$

At this level we admit the value $+\infty$.

- Existence follows from the direct method. From $F(u_n) \leq M$ we obtain a unif bound on $\|\dot{u}_n\|_{L^2}$, and also a unif. bound on $\|u_n\|_{L^\infty}$ due to the DBCs. In the usual way

  $\dot{u}_{n_k} \longrightarrow \dot{u}_\infty$ weakly L^2 and $u_{n_k} \to u_\infty$ unif.

  LSC follows from LSC of the norm wrt to weak convergence and from Fatou's lemma for the other term (we cannot simply exploit unif. conv. because u is in the denominator).

  We observe also that DBCs and condition $u \geq 0$ are stable wrt unif. convergence.
- Uniqueness follows in the usual way (every pos. sol. is a min point, and the minimizer is unique due to ...)
- Truncation argument $\Rightarrow u(x) \geq 1$ in $[0,2]$ (this answers also point (b)).
- Regularity. Once we know that $u(x)$ is bounded away from 0 we can compute ELE as always, obtaining that

  $$(\dot{u})' = -\frac{x^3}{u^3} \qquad \text{(weak derivative of weak derivative)}$$

  Now the usual bootstrap argument leads to $u \in C^\infty$.

<u>Remarks</u>

- Before the truncation argument we are not allowed to compute $F(u+tv)$ for a large enough class of test functions.
- For classical solutions point (b) follows also from a concavity argument.

3. Let us set, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^\ell \arctan\left(\dot{u}^2 - u^2\right) dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0 \right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(c) Determine the infimum as a function of ℓ.

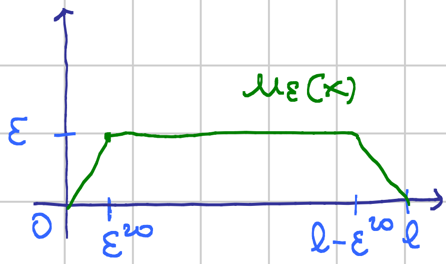
(a) $\boxed{\ell \in (0,\pi)}$ The second variation is $Q(v) = 2\int \dot{v}^2 - v^2$

- If $\ell \in (0,\pi)$ then $u_0(x)$ satisfies $(E) + (L^+) + (J^+)$, and therefore it is a WLM.
- If $\ell > \pi$ then $u_0(x)$ satisfies (L^+) but not (J), and therefore it cannot be a WLM.
- If $\ell = \pi$, assume that $u_0(x)$ is a WLM. Then $\varepsilon \sin x$ would be a WLM as well for ε small, but this is not possible because it does not satisfy (E) (of course some computations are needed)

(b) $u_0(x)$ is $\boxed{\text{NEVER}}$ a SLM.

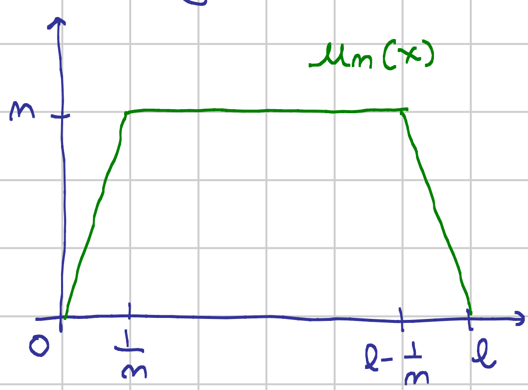
A better candidate in any neighborhood of $u_0(x)$ is shown in the figure.

Indeed $F(u_\varepsilon) = -\ell\varepsilon^2 + o(\varepsilon^2)$.



(c) The infimum is $\boxed{-\frac{\pi}{2}\ell}$ for every $\ell > 0$

A possible minimizing sequence is show below.



Remark In both cases we can obtain smooth sequences by "rounding off" the corners.

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4. For every real number $m > 0$, let us set

$$J(m) := \inf\left\{ \underbrace{\int_0^1 (u^{19} - \sin(u^2))\, dx}_{F(u)} : u \in C^1_c((0,1)),\ \int_0^1 |\dot u|^7\, dx \le m \right\}.$$

(a) Determine whether there exists $m > 0$ such that $J(m) = 0$.

(b) Determine for which real values of α it turns out that

$$\lim_{m\to 0^+} \frac{J(m)}{m^\alpha} = 0.$$

(c) Determine for which real values of β it turns out that

$$\lim_{m\to +\infty} \frac{J(m)}{m^\beta} = 0.$$

(a) $\boxed{\text{NO}}$ Take any function $u_0 \in C^1_c((0,1))$ with $u_0 \not\equiv 0$, and consider $u_\varepsilon(x) = \varepsilon u_0(x)$. When $\varepsilon > 0$ is small enough it turns out that

$$F(u_\varepsilon) = -\varepsilon^2 \int_0^1 u_0(x)^2 + o(\varepsilon^2) < 0 \quad \text{and} \quad \int_0^1 |\dot u_\varepsilon|^7\, dx \le m.$$

Therefore $J(m) \le F(u_\varepsilon) < 0$.

(b) $\boxed{\alpha < \frac{2}{7}}$ With the variable change $u = \varepsilon^{1/7} v$ we obtain that

$$J(m) = m^{2/7} \underbrace{\inf\left\{ \int m^{\frac{17}{7}} v^{19} - \frac{\sin(m^{2/7} v^2)}{m^{2/7}} : \int |\dot v|^7 \le 1 \right\}}$$

$$\to -\inf\left\{ \int v^2 : \int |\dot v|^7 \le 1 \right\} \in (-\infty, 0)$$

(easy direct method)

The verification of Γ-convergence is standard

→ the first term tends to 0 uniformly

→ the second one is bounded from below by $-v^2$

(c) $\boxed{\alpha > \frac{19}{7}}$ The same variable change as before gives that

$$J(m) = m^{\frac{19}{7}} \underbrace{\inf\left\{ \int v^{19} - \frac{\sin(m^{\frac{2}{7}} v^2)}{m^{19/7}} : \int |\dot v|^7 \le 1 \right\}}$$

$$\to \inf\left\{ \int v^{19} : \int |\dot v|^7 \le 1 \right\} \in (0, +\infty)$$

The verification of Γ-convergence is even simpler (actually in this case we have uniform convergence).

Remark In the Γ-conv. results we have extended the functionals to $+\infty$ for all functions $u \in L^2$ that are not in C^1_c and/or do not satisfy the integral constraint.

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